Math178 SU19
Homework 1
Due: Wed, May 29, 2019

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

Note: You need to create a Github account for submission of the coding part of the homework. Please create a repository on Github to hold all your code and include your Github account username as part of the answer to the coding problems.

1 (**Covariance. (Lecture 1 page 17)**) The covariance between two random variables X and Y is defined as:

$$\mathrm{cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

Prove that

$$\mathrm{cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Solution:

$$\begin{aligned}
\mathrm{cov}[X, Y] &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\
&= \mathbb{E}[XY - \mathbb{E}[X]Y - X\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y]] \\
&= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]
\end{aligned}$$

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2 (**Correlation. (Lecture 1 page 18)**) The correlation between two random variables X and Y is defined as:

$$\mathrm{corr}[X,Y] = \frac{\mathrm{cov}[X,Y]}{\sqrt{\mathrm{var}[X]\mathrm{var}[Y]}}.$$

Prove that

(a) $-1 \le \mathrm{corr}[X,Y] \le 1$;

(b) $\mathrm{corr}[X,Y] = 1$ if and only if $Y = aX + b$ for some parameters $a \neq 0$ and b.

Solution:

(a)

$$\begin{aligned} 0 &\le \mathrm{var}\left[\frac{X}{\sqrt{\mathrm{var}[X]}} - \frac{Y}{\sqrt{\mathrm{var}[Y]}}\right] \\ &= \mathrm{var}\left[\frac{X}{\sqrt{\mathrm{var}[X]}}\right] + \mathrm{var}\left[\frac{Y}{\sqrt{\mathrm{var}[Y]}}\right] - 2\mathrm{cov}\left[\frac{X}{\sqrt{\mathrm{var}[X]}}, \frac{Y}{\sqrt{\mathrm{var}[Y]}}\right] \\ &= 2 - 2\mathrm{corr}[X,Y] \end{aligned}$$

gives $\mathrm{corr}[X,Y] \le 1$. Likewise $0 \le \mathrm{var}\left[\frac{X}{\sqrt{\mathrm{var}[X]}} + \frac{Y}{\sqrt{\mathrm{var}[Y]}}\right]$ gives $\mathrm{corr}[X,Y] \ge -1$.

(b) $\mathrm{corr}[X,Y] = 1$ if and only if $\mathrm{var}\left[\frac{X}{\sqrt{\mathrm{var}[X]}} - \frac{Y}{\sqrt{\mathrm{var}[Y]}}\right] = 0$, which implies $\frac{X}{\sqrt{\mathrm{var}[X]}} - \frac{Y}{\sqrt{\mathrm{var}[Y]}} = C$ for some constant C.

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3 (Parametrization. (Lecture 1 page 50)) Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Solution:
Let $l(t) = ||\alpha(t)||^2$ be the distance from point t on the curve to the origin. To minimize $l(t)$ is equivalent to minimize $||\alpha(t)||$. Suppose $l(t)$ achieve its minimum at t_0, then $l'(t_0) = 0$. Now

$$l(t) = \alpha(t) \cdot \alpha(t)$$

$$l'(t) = \alpha'(t) \cdot \alpha(t) + \alpha(t) \cdot \alpha'(t) = 2\alpha'(t) \cdot \alpha(t) = 0$$

implies $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. Here $\cdot$ is the dot product. ∎

4 (**Extra credit. (Lecture 1 page 52)**) How to create a transformation from the data on some helix to the data of the instructors trajectory?

See hw 1 code solution. ■

5 (**Coding. (Lecture 1 page 54-70)**) Please download the H-MOG dataset from: http://www.cs.wm.edu/ qyang/hmog.html (see also Lecture 1 page 54). Please read through the data description and do some visualizations if you have time. If you have any visualization result, please email or print out to submit.

See hw 2 code solution. ■