Math178 SU19
Homework 3
Due: Mon, June 3, 2019

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

Note: You need to create a Github account for submission of the coding part of the homework. Please create a repository on Github to hold all your code and include your Github account username as part of the answer to the coding problems.

1 (**Regular Surface.**) Show that a sphere (S^2) is a regular surface using spherical coordinates.

Solution:
A sphere $(S^2) = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^3 = 1\}$ can be parametrized by using spherical coordinates. Let $V_1 = \{(\theta, \psi) | 0 < \theta < \pi, 0 < \psi < 2\pi\}$ and a parametrization $\mathbf{x}_1 : V_1 \to \mathbb{R}^3$ is given by

$$\mathbf{x}_1(\theta, \psi) = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta).$$

This parametrization covers almost every part of the sphere (except the north and south poles, and a half great circle connecting them). In order to cover the whole sphere, we need more parametrizations, such as let $V_2 = \{(\theta, \psi) | -\pi/2 < \theta < \pi/2, 0 < \psi < 2\pi\}$ and a parametrization $\mathbf{x}_2 : V_2 \to \mathbb{R}^3$ such that

$$\mathbf{x}_2(\theta, \psi) = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta).$$

To prove that $\mathbf{x}_1$ satisfies the three conditions (similar for $\mathbf{x}_2$):

(a) It is clear that the functios $\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta$ have continuous partial derivatives of all orders; hence, $\mathbf{x}_1$ is differentiable.

(b) Given $(x, y, z) \in S^2 - C$, where C is the semicircle

$$C = \{(x, y, z) \in S^2; y = 0, x \geq 0\},$$

θ is uniquely determined by $\theta = \cos^{-1} z$, since $0 < \theta < \pi$. By knowing θ, we find $\sin\psi$ and $\cos\psi$ from $x = \sin\theta\cos\psi, y = \sin\theta\sin\psi$, and this determines ψ uniquely ($0 < \psi < 2\pi$). It follows that $\mathbf{x}_1$ has an inverse $\mathbf{x}_1^{-1}$.

(c) In order that the Jacobian determinants

$$\frac{\partial(x, y)}{\partial(\theta, \psi)} = \cos\theta\sin\theta,$$

$$\frac{\partial(y,z)}{\partial(\theta,\psi)} = \sin^2\theta\cos\psi,$$

$$\frac{\partial(x,z)}{\partial(\theta,\psi)} = \sin^2\theta\sin\psi$$

vanish simultaneously, it is necessary that

$$\cos^2\theta\sin^2\theta + \sin^4\theta\cos^2\psi + \sin^4\theta\sin^2\psi = \sin^2\theta = 0.$$

This does not happen in V_1, and condition 3 is satisfied.

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2 (**Short Cut 2 of Regular Surface.**) Show Torus (T^2) is a regular surface using short cut 2.

See lecture 3 slides page 24-25. ∎