Math178 SU19
Homework 4
Due: Mon, June 10, 2019

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

Note: You need to create a Github account for submission of the coding part of the homework. Please create a repository on Github to hold all your code and include your Github account username as part of the answer to the coding problems.

1 (**Rigid Body.**) Recall that

$$SE(3) = \left\{ A \,\middle|\, A = \begin{bmatrix} R & \vec{v} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix},\ R \in \mathbb{R}^{3\times 3}, \vec{v} \in \mathbb{R}^3, R^T R = RR^T = I, \det|R| = 1 \right\},$$

where R is rotation and $\vec{v}$ is translation. Show that $SE(3)$ is a group.

Solution:
Assume that $A_i = \begin{bmatrix} R_i & \vec{v}_i \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \in SE(3)$ satisfying $R_i \in \mathbb{R}^{3\times 3}, \vec{v}_i \in \mathbb{R}^3, R_i^T R_i = R_i R_i^T = I, \det|R_i| = 1$, for $i = 1,2,3$. We need to verify the following:

- Show $A_1 A_2 \in SE(3)$:

$$A_1 A_2 = \begin{bmatrix} R_1 & \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} R_2 & \vec{v}_2 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1\vec{v}_2 + \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}.$$

  Since $R_1 R_2 \in \mathbb{R}^{3\times 3}, R_1\vec{v}_2 + \vec{v}_1 \in \mathbb{R}^3$,

$$(R_1 R_2)^T R_1 R_2 = R_2^T R_1 R_1^T R_2^T = I,$$

$$\det|R_1 R_2| = \det|R_1|\det|R_2| = 1,$$

  we verified $A_1 A_2 \in SE(3)$.

- Show $(A_1 A_2)A_3 = A_1(A_2 A_3)$: Since

$$(A_1 A_2)A_3 = \begin{bmatrix} R_1 R_2 & R_1\vec{v}_2 + \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} R_3 & \vec{v}_3 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 R_3 & R_1 R_2\vec{v}_3 + R_1\vec{v}_2 + \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

  and

$$A_1(A_2 A_3) = \begin{bmatrix} R_1 & \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} R_2 R_3 & R_2\vec{v}_3 + \vec{v}_2 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 R_3 & R_1 R_2\vec{v}_3 + R_1\vec{v}_2 + \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix},$$

  we verified $(A_1 A_2)A_3 = A_1(A_2 A_3)$.

- Exist identity element $I \in SE(3)$ such that $A_1 I = A_1$: Let $I \in SE(3) = \begin{bmatrix} I_{3\times 3} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$ with I be the identity matrix in $\mathbb{R}^{3\times 3}$.

$$A_1 I = \begin{bmatrix} R_1 & \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} I_{3\times 3} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} R_1 & \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = A_1.$$

- Exist identity inverse $A_1^{-1} \in SE(3)$ such that $A_1 A_1^{-1} = I$: Let $A_1^{-1} \in SE(3) = \begin{bmatrix} R_1^T & -R_1^T \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$.

$$A_1 A_1^{-1} = \begin{bmatrix} R_1 & \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} R_1^T & -R_1^T \vec{v}_1 \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} = I.$$

■