Math178 SU19
Homework 5
Due: Mon, June 17, 2019

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

Note: You need to create a Github account for submission of the coding part of the homework. Please create a repository on Github to hold all your code and include your Github account username as part of the answer to the coding problems.

1 (**Christoffel Symbol.**) Compute the Christoffel symbols for a surface of revolution parametrized by

$$\mathbf{x}(u,v) = (f(v)\cos(u), f(v)\sin(u), g(v)), \;\; f(v) \neq 0.$$

Solution:
Take derivative of $\mathbf{x}$ with respect to u, v:

$$\mathbf{x}_u = (-f(v)\sin(u), f(v)\cos(u), 0)$$

$$\mathbf{x}_v = (f'(v)\cos(u), f'(v)\sin(u), g'(v)).$$

Then take derivative of $\mathbf{x}_u$ with respect to u, v:

$$\mathbf{x}_{uu} = (-f(v)\cos(u), -f(v)\sin(u), 0)$$

$$\mathbf{x}_{uv} = (-f'(v)\sin(u), f'(v)\cos(u), 0).$$

$$\mathbf{x}_{vv} = (f''(v)\cos(u), f''(v)\sin(u), g''(v)).$$

Now we can calculate

$$E =< \mathbf{x}_u, \mathbf{x}_u >= f^2(v),$$

$$F =< \mathbf{x}_u, \mathbf{x}_v >= 0,$$

$$G =< \mathbf{x}_v, \mathbf{x}_v >= (f'(v))^2 + (g'(v))^2,$$

then

$$E_u = 0, \; F_u = 0, \; G_u = 0,$$

$$E_v = 2f(v)f'(v), \; F_v = 0, \; G_v = 2f'(v)f''(v) + 2g'(v)g''(v).$$

To solve the Christoffel symbols, recall the systems of Christoffel symbols:

$$\begin{cases} E\Gamma^1_{11} + F\Gamma^2_{11} = E_u/2 \\ F\Gamma^1_{11} + G\Gamma^2_{11} = F_u - E_v/2 \end{cases}$$

$$\begin{cases} E\Gamma^1_{12} + F\Gamma^2_{12} = E_v/2 \\ F\Gamma^1_{12} + G\Gamma^2_{12} = G_u/2 \end{cases}$$

$$\begin{cases} E\Gamma^1_{22} + F\Gamma^2_{22} = F_v - G_u/2 \\ F\Gamma^1_{22} + G\Gamma^2_{22} = G_v/2 \end{cases}$$

Now we have

$$\Gamma^1_{11} = \frac{\begin{vmatrix} E_u/2 & F \\ F_u - E_v/2 & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \frac{\begin{vmatrix} 0 & 0 \\ -f(v)f'(v) & (f'(v))^2 + (g'(v))^2 \end{vmatrix}}{\begin{vmatrix} f^2(v) & 0 \\ 0 & (f'(v))^2 + (g'(v))^2 \end{vmatrix}} = 0$$

$$\Gamma^2_{11} = \frac{\begin{vmatrix} E & E_u/2 \\ F & F_u - E_v/2 \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \frac{\begin{vmatrix} f^2(v) & 0 \\ 0 & -f(v)f'(v) \end{vmatrix}}{\begin{vmatrix} f^2(v) & 0 \\ 0 & (f'(v))^2 + (g'(v))^2 \end{vmatrix}} = \frac{-f(v)f'(v)}{(f'(v))^2 + (g'(v))^2}$$

$$\Gamma^1_{12} = \frac{\begin{vmatrix} E_v/2 & F \\ G_u/2 & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \frac{\begin{vmatrix} f(v)f'(v) & 0 \\ 0 & (f'(v))^2 + (g'(v))^2 \end{vmatrix}}{\begin{vmatrix} f^2(v) & 0 \\ 0 & (f'(v))^2 + (g'(v))^2 \end{vmatrix}} = \frac{f'(v)}{f(v)}$$

$$\Gamma^2_{12} = \frac{\begin{vmatrix} E & E_v/2 \\ F & G_u/2 \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \frac{\begin{vmatrix} f^2(v) & f(v)f'(v) \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} f^2(v) & 0 \\ 0 & (f'(v))^2 + (g'(v))^2 \end{vmatrix}} = 0$$

$$\Gamma^1_{22} = \frac{\begin{vmatrix} F_v - G_u/2 & F \\ G_v/2 & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \frac{\begin{vmatrix} 0 & 0 \\ f'(v)f''(v) + g'(v)g''(v) & (f'(v))^2 + (g'(v))^2 \end{vmatrix}}{\begin{vmatrix} f^2(v) & 0 \\ 0 & (f'(v))^2 + (g'(v))^2 \end{vmatrix}} = 0$$

$$\Gamma^2_{22} = \frac{\begin{vmatrix} E & F_v - G_u/2 \\ F & G_v/2 \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \frac{\begin{vmatrix} f^2(v) & 0 \\ 0 & f'(v)f''(v) + g'(v)g''(v) \end{vmatrix}}{\begin{vmatrix} f^2(v) & 0 \\ 0 & (f'(v))^2 + (g'(v))^2 \end{vmatrix}} = \frac{f'(v)f''(v) + g'(v)g''(v)}{(f'(v))^2 + (g'(v))^2}$$

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