Math178 SU19
Homework 6
Due: Fri, June 21, 2019

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

Note: You need to create a Github account for submission of the coding part of the homework. Please create a repository on Github to hold all your code and include your Github account username as part of the answer to the coding problems.

1 (1st fundamental form.) Compute the first fundamental form of a sphere at a point of the coordinate neighborhood given by the parametrization:

$$\mathbf{x}(\theta, \psi) = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta).$$

Solution:
First observe that

$$\mathbf{x}_\theta(\theta, \psi) = (\cos\theta\cos\psi, \cos\theta\sin\psi, -\sin\theta)$$
$$\mathbf{x}_\psi(\theta, \psi) = (-\sin\theta\sin\psi, \sin\theta\cos\psi, 0).$$

Hence,

$$E(\theta, \psi) =< \mathbf{x}_\theta, \mathbf{x}_\theta >= 1$$
$$F(\theta, \psi) =< \mathbf{x}_\theta, \mathbf{x}_\psi >= 0$$
$$G(\theta, \psi) =< \mathbf{x}_\psi, \mathbf{x}_\psi >= \sin^2\theta.$$

Thus, if w is a tangent vector to the sphere at the point $\mathbf{x}(\theta, \psi)$, given in the basis associated to $\mathbf{x}(\theta, \psi)$ by

$$w = a\mathbf{x}_\theta + b\mathbf{x}_\psi,$$

then the square of the length of w is given by

$$|w|^2 = I(w) =< w, w >=< a\mathbf{x}_\theta + b\mathbf{x}_\psi, a\mathbf{x}_\theta + b\mathbf{x}_\psi >$$

$$=< \mathbf{x}_\theta, \mathbf{x}_\theta > a^2 + 2 < \mathbf{x}_\theta, \mathbf{x}_\psi > ab + < \mathbf{x}_\psi, \mathbf{x}_\psi > b^2$$
$$= Ea^2 + 2Fab + Gb^2 = a^2 + \sin^2\theta b^2.$$

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2 (**Area.**) Compute the area of the torus with the coordinate neighborhood corresponding to the parametrization:

$$\mathbf{x}(u,v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u),\ 0 < u < 2\pi, 0 < v < 2\pi,$$

which covers the torus, except for a meridian and a parallel.

Solution:
Since

$$\mathbf{x}_u(u,v) = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$$
$$\mathbf{x}_v(u,v) = (-(a + r\cos u)\sin v, (a + r\cos u)\cos v, 0),$$

the coefficients of the first fundamental form are

$$E =< \mathbf{x}_u, \mathbf{x}_u >= r^2$$

$$F =< \mathbf{x}_u, \mathbf{x}_v >= 0$$

$$G =< \mathbf{x}_v, \mathbf{x}_v >= (a + r\cos u)^2;$$

hence,

$$\sqrt{EG - F^2} = r(a + r\cos u).$$

Now, let $\epsilon > 0$ and small. Consider the region R_ϵ obtained as the image by $\mathbf{x}$ of the region Q_ϵ (Lecture 8 Part 1 Page 24) given by

$$Q_\epsilon = \{(u,v) \in \mathbb{R}^2; 0 + \epsilon \le u \le 2\pi - \epsilon, 0 + \epsilon \le v \le 2\pi - \epsilon\}.$$

Use the definition of area (Lecture 8 Part 1 Page 22), we obtain

$$A(R_\epsilon) = \int\int_{Q_\epsilon} r(a + r\cos u)dudv = \int_{0+\epsilon}^{2\pi-\epsilon} r(a + r\cos u)du \int_{0+\epsilon}^{2\pi-\epsilon} dv$$

$$= r^2(2\pi - 2\epsilon)(\sin(2\pi - \epsilon) - \sin\epsilon) + ra(2\pi - 2\epsilon)^2.$$

Letting $\epsilon \to 0$ in the above expression, we obtain

$$A(T) = \lim_{\epsilon \to 0} A(R_\epsilon) = 4\pi^2 ra.$$

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3 (**Coding.**) Please use the 6 features (accelerometer: x, y, z and gyroscope: x, y, z) of H-MOG dataset to do the following:

(a) Pick some users. For each user pick 3 out of the 6 features. (Or if you have time, you can try all the 20 combinations.)

(b) For each data point of the 3 features v_1, v_2, v_3, normalize the vector $\vec{v} = [v_1, v_2, v_3]$ by:

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||_2}.$$

(c) Plot the normalized data points (vectors) on a sphere.

Note that a starter file is included under "resource" tab. Please feel free to ask TA if you have any question.

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