Lecture 3

Math 178

Nonlinear Data Analytics

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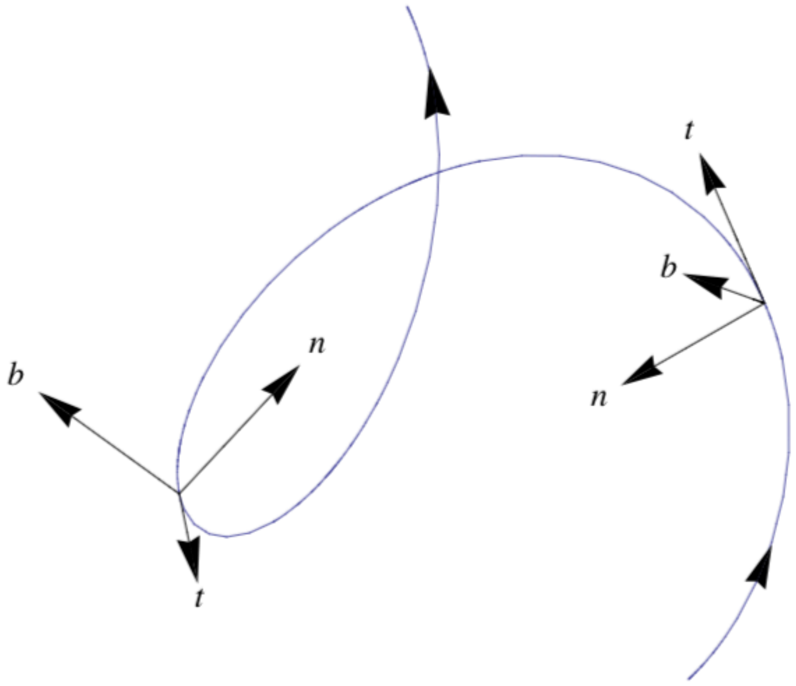
Last time: Torsion

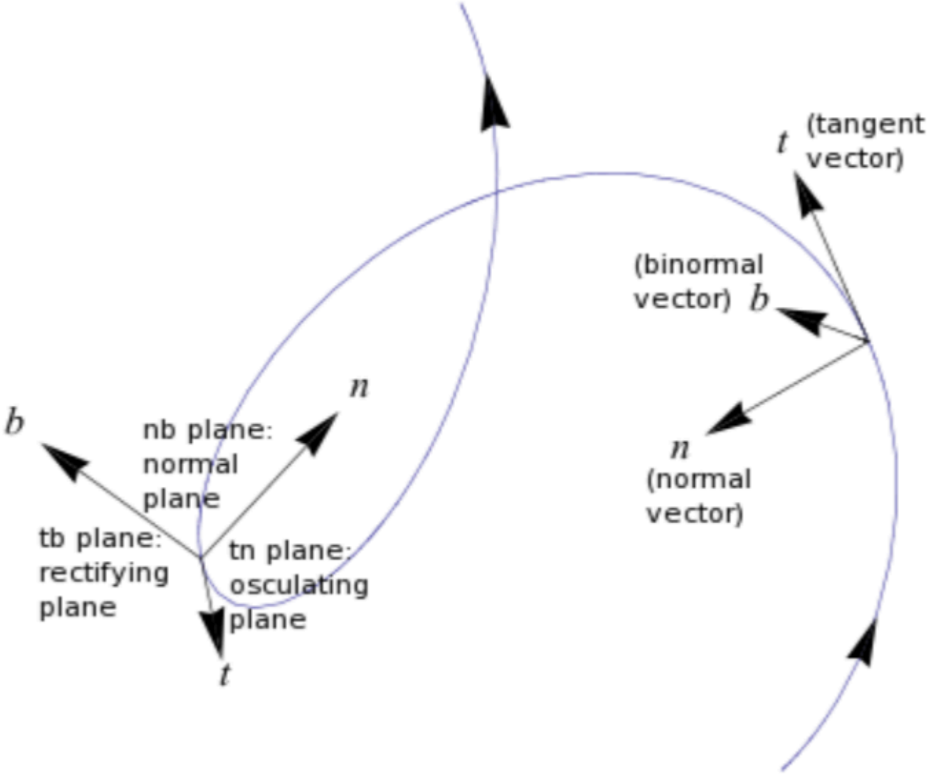
Geometric Meaning

Since $b(s)$ is a unit vector, the length $\|b'(s)\|$ measures the rate of change of the neighboring osculating planes with the osculating plane at s; that is $b'(s)$ measures how rapidly the curve pulls away from the osculating plane at s, in a neighborhood of s.



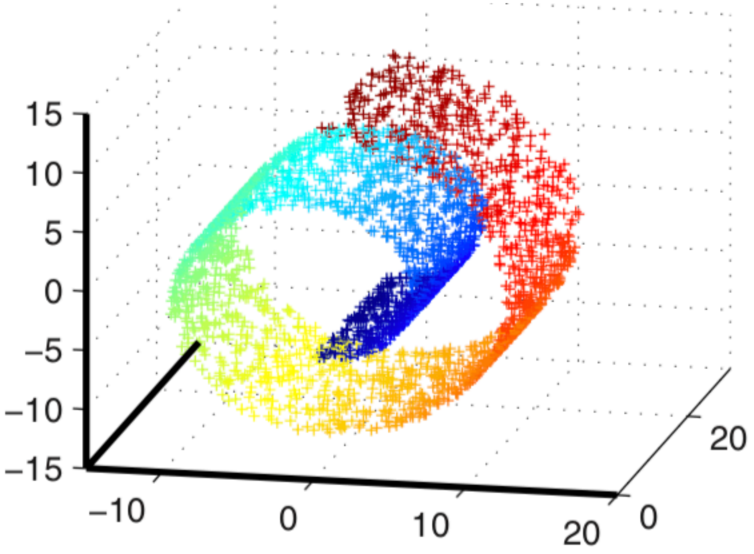
Frenet Frame

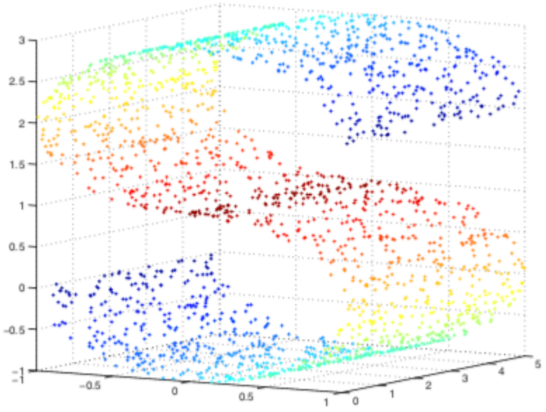
$$\alpha'(s) \stackrel{\triangle}{=} t(s)$$
$$\alpha''(s) = k(s)n(s)$$
$$t(s) \wedge n(s) = b(s)$$

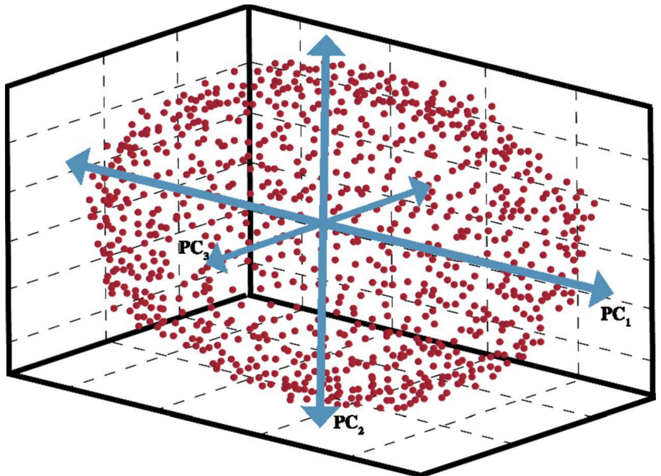


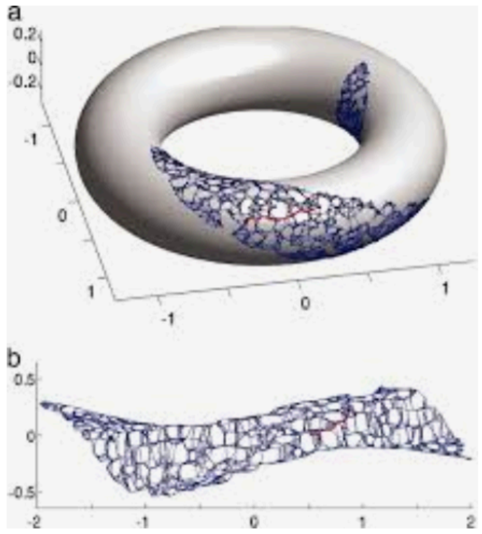
Why do we need nonlinear data analytics and why are they important?

- High dimensional data typically lives on or is near a low-dimensional manifold, but that manifold is not necessarily -- and usually not – linear!



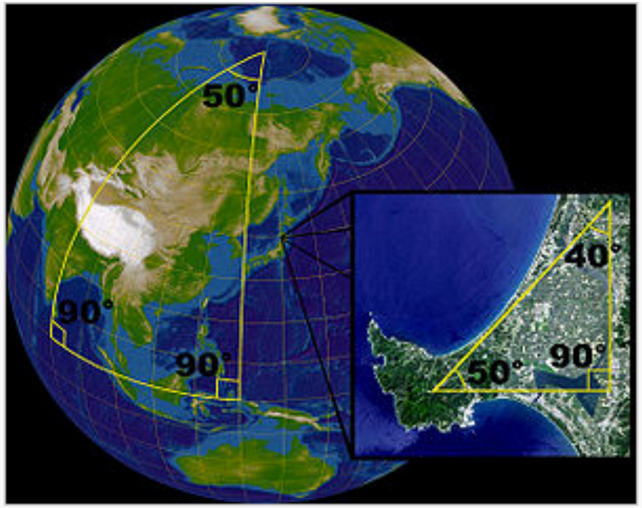




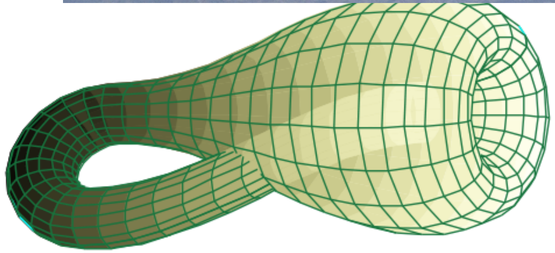


What is a manifold?

- An n-dimensional manifold locally "looks like" a piece of $\mathbf{R}^n$.
- For examples, sphere and torus.
- Key features of a **manifold**: **curved**



The sphere (surface of a ball) is a two-dimensional manifold since it can be represented by a collection of two-dimensional maps.



- Only manifolds can capture UAV's dynamical behaviors

From Regular Surface to Manifold

Definition

A subset $S \subset \mathbb{R}^3$ is a *regular surface* if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. $\mathbf{x}$ is differentiable (so we can use calculus).
2. $\mathbf{x}$ is a homeomorphism (so we can use analysis)
3. $\mathbf{x}$ is regular (so we can use linear algebra)

Remark

In contrast to our treatment of curves, we have *defined a surface as a subset* S of $\mathbb{R}^3$, and not as a map. This is achieved by covering S with the traces of parametrizations which satisfy conditions 1, 2, and 3.

Exact meanings:

$\mathbf{x}$ is differentiable

This means that if we write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U,$$

the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ have continuous partial derivatives of all orders.

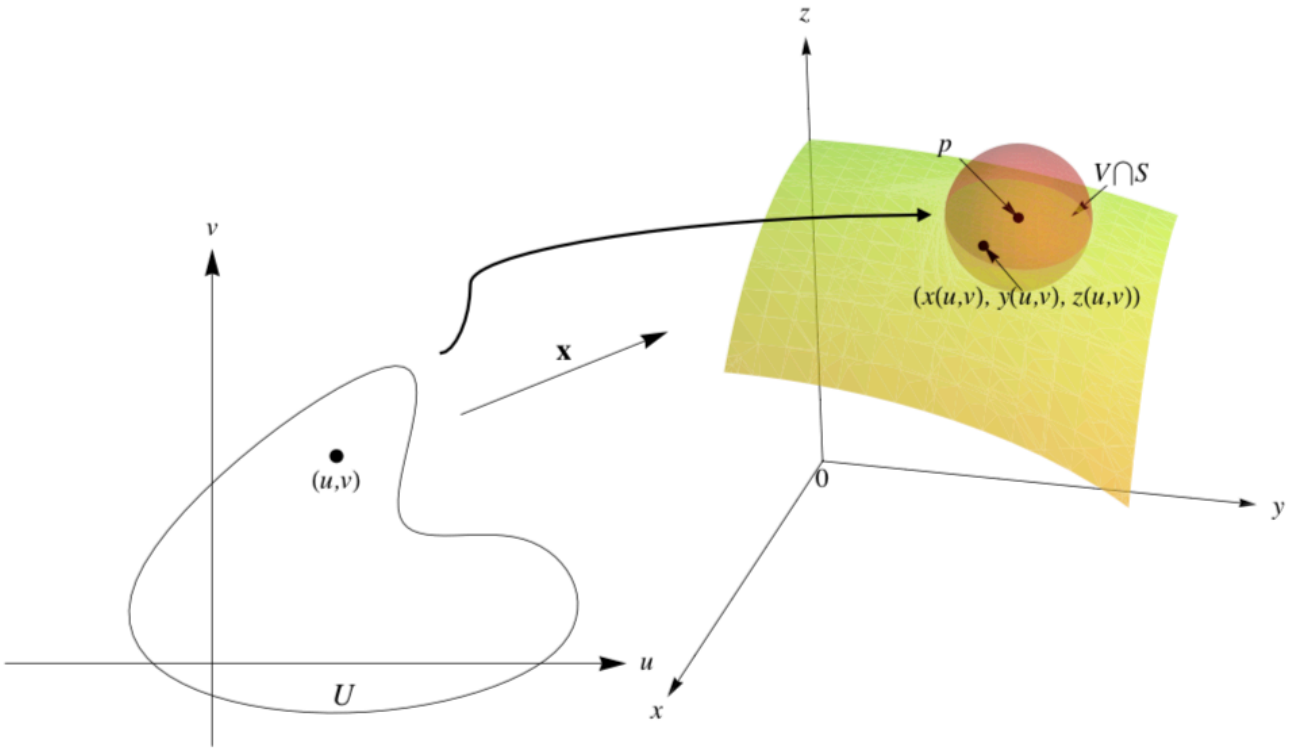
$\mathbf{x}$ is a homeomorphism

Since $\mathbf{x}$ is continuous by condition 1, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous; that is, $\mathbf{x}^{-1}$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \to \mathbb{R}^2$ defined on an open set W containing $V \cap S$.

$\mathbf{x}$ is regular

For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

A Parametrization
and a coordinate neighborhood



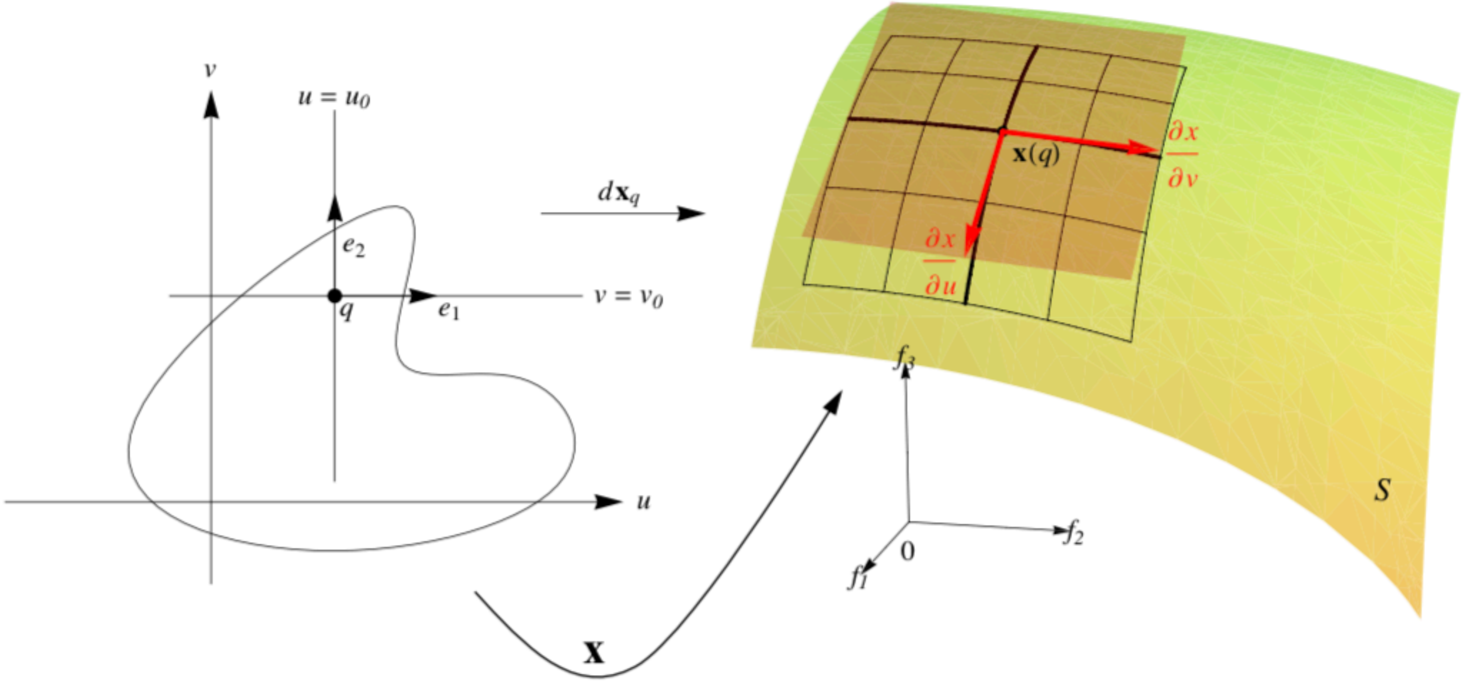
Definition

The mapping $\mathbf{x}$ is called a *parametrization* or a *system of (local) coordinates* in (a neighborhood of) p. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood*.

The Regularity Condition

An Illustrative Example

To give condition 3 a more familiar form, let us compute the matrix of the linear map $d\mathbf{x}_q$ in the canonical bases $e_1 = (1, 0)$, $e_2 = (0, 1)$ of $\mathbb{R}^2$ with coordinates u, v and $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 0, 1)$ of $\mathbb{R}^3$, with coordinates (x, y, z).



The Regularity Condition

An Illustrative Example (cont'd)

Thus, the matrix of the linear map $d\mathbf{x}_q$ in the referred (standard) basis is

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

Condition 3 may now be expressed by requiring the two column vectors of this matrix to be linearly independent; or, equivalently, that the vector product $\partial\mathbf{x}/\partial u \wedge \partial\mathbf{x}/\partial v \neq 0$; or, in still another way, that one of the minors of order 2 of the matrix $d\mathbf{x}_q$, that is, one of the Jacobian determinants

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y,z)}{\partial(u,v)}, \quad \frac{\partial(x,z)}{\partial(u,v)},$$

be nonzero at q.

The Three Conditions

- Condition 1 is very natural if we expect to do some differential geometry on S.
- The one-to-oneness in condition 2 has the purpose of preventing self-intersections in regular surfaces. This is clearly necessary if we are to speak about, say, *the* tangent plane at a point $p \in S$. The continuity of the inverse in condition 2 has a more subtle purpose. For the time being, we shall mention that this condition is essential to proving that certain objects defined in terms of a parametrization do not depend on this parametrization but only on the set S itself.
- Finally, condition 3 will guarantee the existence of a "tangent plane" at all points of S.

Proving that a Set is a Regular Surface

Example

Let us show that the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is a regular surface.

Method 1: Using Cartesian Coordinates

We first verify that the map $\mathbf{x}_1 : U \in \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{x}_1(x, y) = (x, y, +\sqrt{1 - (x^2 + y^2)}), \quad (x, y) \in U,$$

where $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ and
$U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is a parametrization of S^2.

Proving that a Set is a Regular Surface

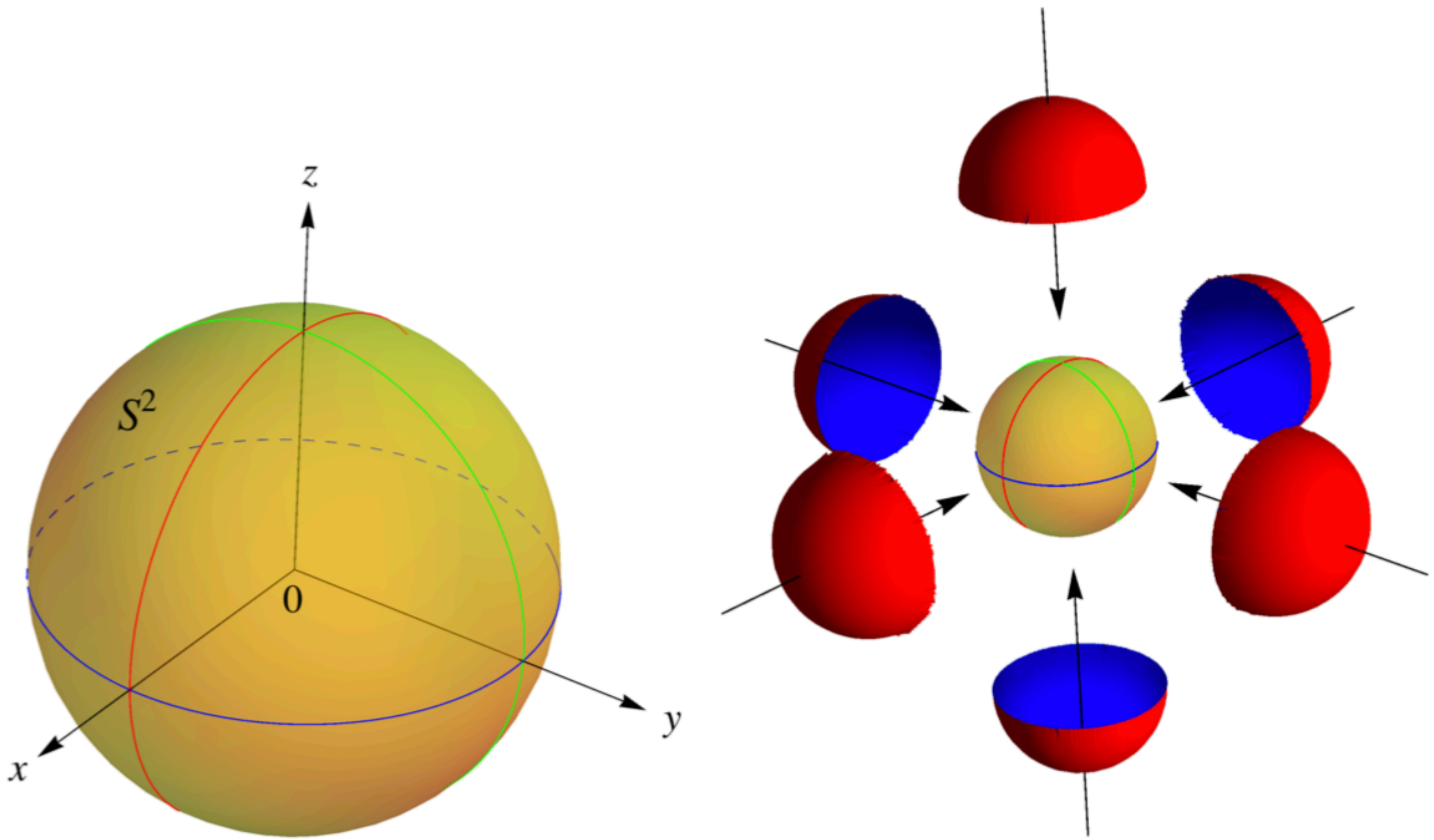
We shall now cover the whole sphere with similar parametrizations as follows. we define $\mathbf{x}_2 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{x}_2(x,y) = (x, y, -\sqrt{1-(x^2+y^2)}),$$

check that $\mathbf{x}_2$ is a parametrization, and observe that $\mathbf{x}_1(U) \cup \mathbf{x}_2(U)$ covers S^2 minus the equator $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=1, z=0\}$. Then, using the xz and zy planes, we define the parametrization

$$\begin{aligned}
\mathbf{x}_3(x,z) &= (x, +\sqrt{1-(x^2+z^2)}, z),\\
\mathbf{x}_4(x,z) &= (x, -\sqrt{1-(x^2+z^2)}, z),\\
\mathbf{x}_5(y,z) &= (+\sqrt{1-(y^2+z^2)}), y, z),\\
\mathbf{x}_6(y,z) &= (-\sqrt{1-(y^2+z^2)}), y, z),
\end{aligned}$$

which, together with $\mathbf{x}_1$ and $\mathbf{x}_2$, cover S^2 completely and shows that S^2 is a regular surface.


z
S^2
0
x
y

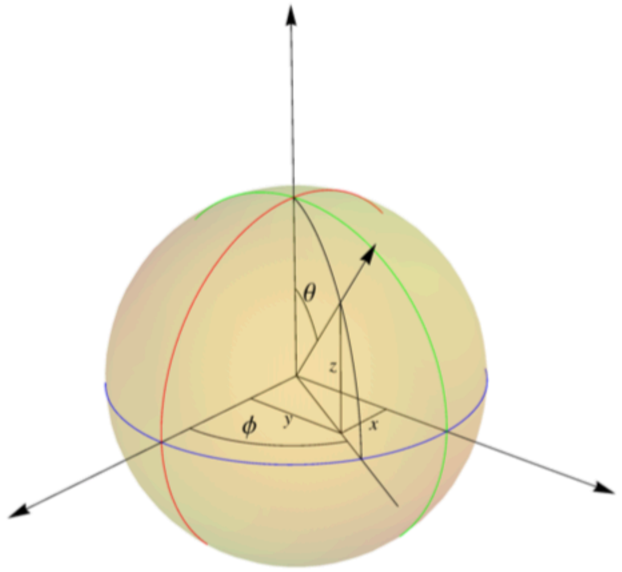
Proving that a Set is a Regular Surface

Method 2: Using Spherical Coordinates

For most applications, it is convenient to relate parametrizations to the geographical coordinates on S^2. Let $V = \{(\theta, \varphi) \mid 0 < \theta < \pi, 0 < \varphi < 2\pi\}$ and let $\mathbf{x} : V \to \mathbb{R}^3$ be given by

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Clearly, $\mathbf{x}(V) \subset S^2$.



Proving that a Set is a Regular Surface

We shall prove that $\mathbf{x}$ is a parametrization of S^2.

Next, we observe that given $(x, y, z) \in S^2 \setminus C$, where C is the semicircle $C = \{(x, y, z) \in S^2 \mid y = 0, x \geq 0\}$, θ is uniquely determined by $\theta = \cos^{-1} z$, since $0 < \theta < \pi$. By knowing θ, we find $\sin\varphi$ and $\cos\varphi$ from $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, and this determines φ uniquely $(0 < \varphi < 2\pi)$. It follows that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1}$. To complete the verification of condition 2, we should prove that $\mathbf{x}^{-1}$ is continuous. However, since we shall soon prove that this verification is not necessary provided we already know that the set S is a regular surface, we shall not do that here.

We remark that $\mathbf{x}(V)$ only omits a semicircle of S^2 (including the two poles) and that S^2 can be covered with the coordinate neighborhoods of two parametrizations of this type.

Two Shortcuts

The last example in the previous lecture shows that deciding whether a given subset of $\mathbb{R}^3$ is a regular surface directly from the definition may be quite tiresome.

Shortcut 1

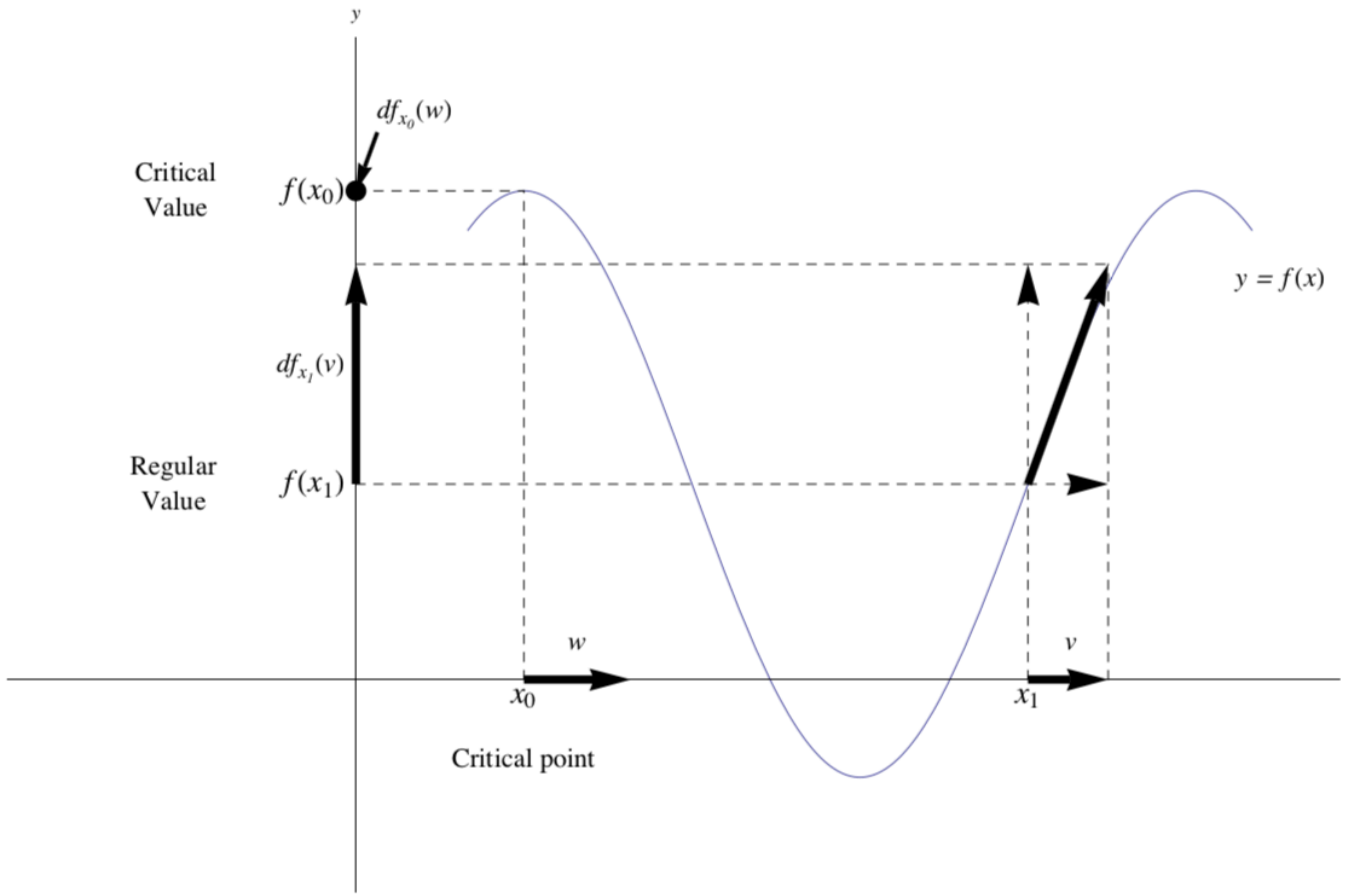
If $f : U \to \mathbb{R}$ is a differentiable function in an open set U of $\mathbb{R}^2$, then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$, is a regular surface

Critical Points and Values

Definition

Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$ we say that $p \in U$ is a *critical point* of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective (or onto) mapping. The image $F(p) \in \mathbb{R}^m$ of a critical point is called a *critical value* of F. A point of $\mathbb{R}^m$ which is not a critical value is called a *regular value* of F.

The terminology is evidently motivated by the particular case in which $f : U \subset \mathbb{R} \to \mathbb{R}$ is a real-valued function of a real variable. A point $x_0 \in U$ is critical if $f'(x_0) = 0$, that is, if the differential df_{x_0} carries all the vectors in $\mathbb{R}$ to the zero vector. Notice that any point $a \notin f(U)$ is trivially a regular value of f.



y
$df_{x_0}(w)$
Critical
Value
$f(x_0)$
$y = f(x)$
$df_{x_1}(v)$
Regular
Value
$f(x_1)$
w
v
x_0
x_1
Critical point

Critical Points and Values

Remark

If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, then

$$df_p = (f_x, f_y, f_z).$$

Note, in this case, that to say that df_p is not surjective is equivalent to saying that $f_x = f_y = f_z = 0$ at p. Hence, $a \in f(U)$ is a regular value of $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ if and only if f_x, f_y, and f_z do not vanish simultaneously at any point in the inverse image

$$f^{-1}(a) = \{(x, y, z) \in U \mid f(x, y, z) = a\}.$$

Two Shortcuts

Shortcut 2

If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Examples

Example

The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

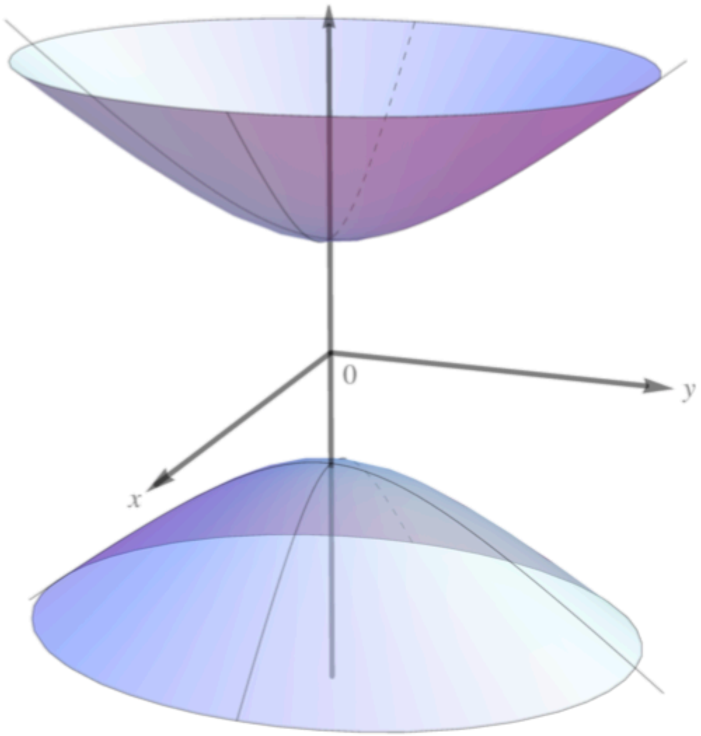
is a regular surface.

The examples of regular surfaces presented so far have been connected subsets of $\mathbb{R}^3$. A surface $S \subset \mathbb{R}^3$ if said to be *connected* if any two of its points can be joined by a continuous curve in S. In the definition of a regular surface we made no restrictions on the connectedness of the surfaces, and the following example shows that the regular surfaces given by Shortcut 2 may not be connected.

Examples

Example

The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ is a regular surface. Note that the surface S is not connected.



Examples

Example

The torus T is a "surface" generated by rotating a circle S^1 of radius r about a straight line belonging to the plane of the circle and at a distance $a > r$ away from the center of the circle.

Proof

Let S^1 be the circle in the yz plane with its center on the point $(0, a, 0)$. Then S^1 is given by $(y - a)^2 + z^2 = r^2$.

The points of T are obtained by rotating this circle about the z axis satisfying the equation

$$\left(\sqrt{x^2 + y^2} - a\right)^2 + z^2 = r^2.$$

Examples

Proof (cont'd)

Let $f(x, y, z) = (\sqrt{x^2 + y^2} - a)^2 + z^2$. Then

$$\frac{\partial f}{\partial z} = 2z, \quad \frac{\partial f}{\partial y} = \frac{2y\left(\sqrt{x^2+y^2} - a\right)}{\sqrt{x^2+y^2}}, \quad \frac{\partial f}{\partial x} = \frac{2x\left(\sqrt{x^2+y^2} - a\right)}{\sqrt{x^2+y^2}}.$$

Hence, $(f_x, f_y, f_z) \neq (0, 0, 0)$ in $f^{-1}(r^2)$, so r^2 is a regular value.
Therefore, the torus is a regular surface.

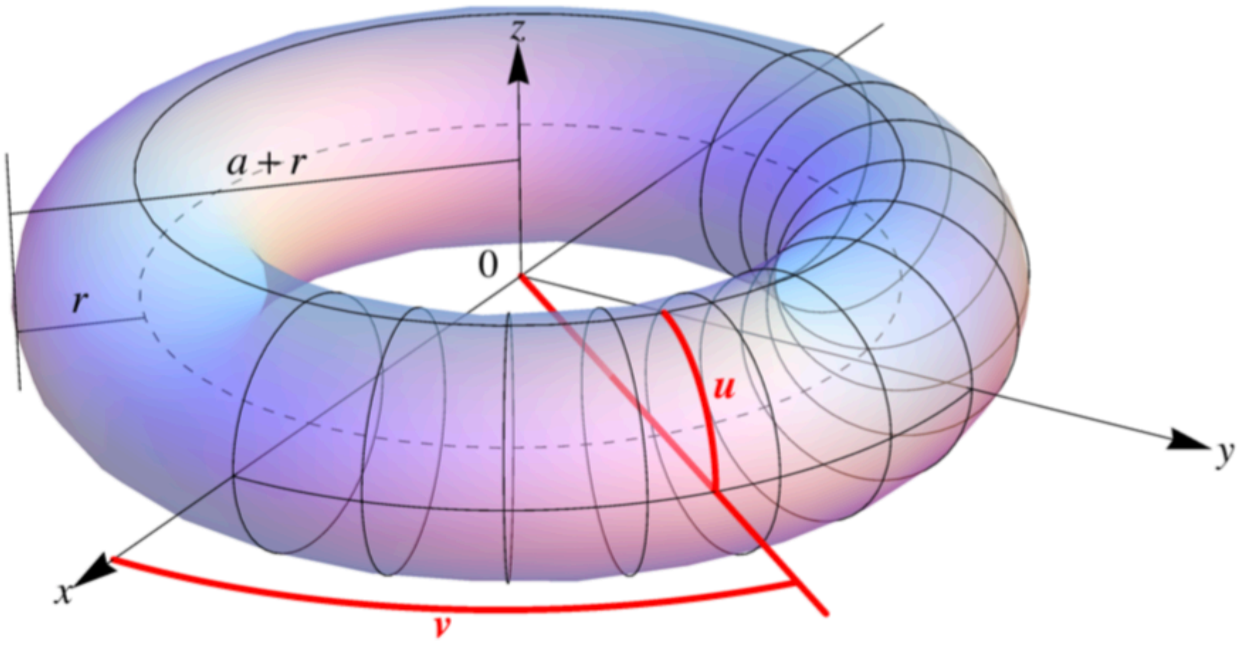
Examples

Example

A parametrization for the torus T of the previous example can be given by

$$\mathbf{x}(u,v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u),$$

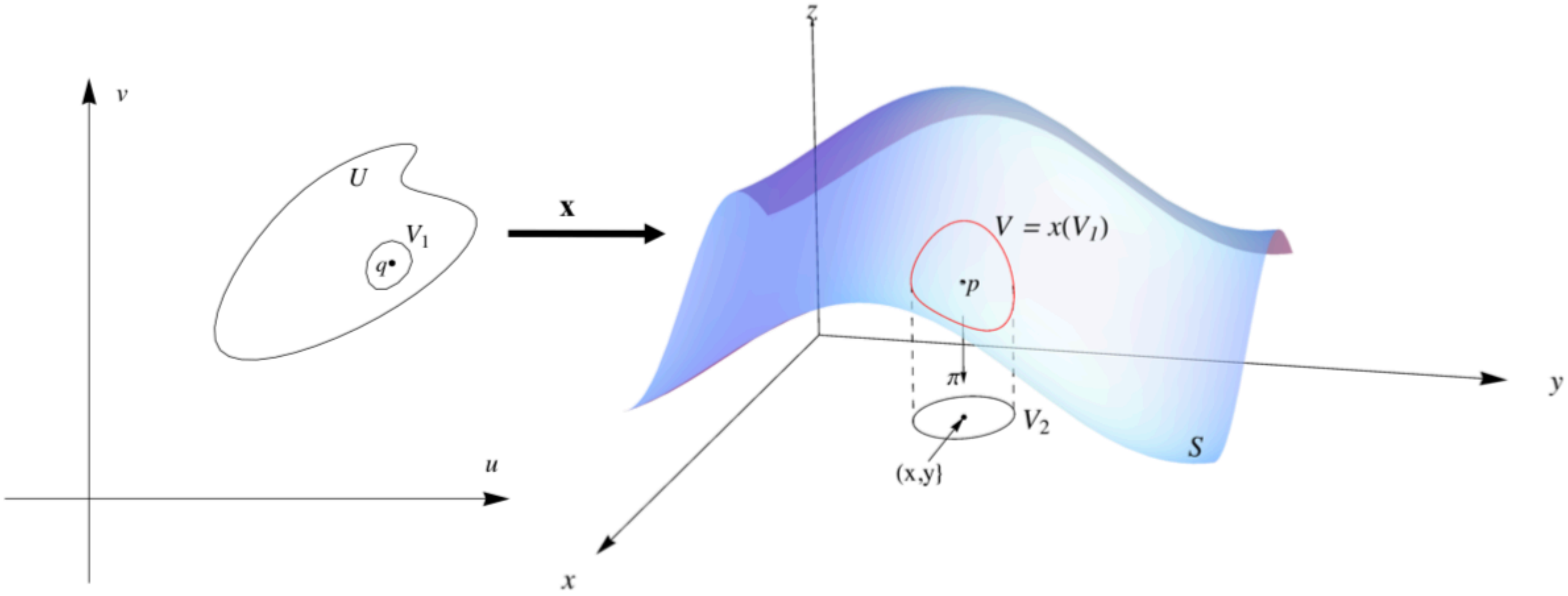
where $0 < u < 2\pi$, $0 < v < 2\pi$.



Two Propositions

Proposition

Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then there exists a neighborhood V of p in S such that V is the graph of a differentiable function which has one of the following three forms $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. (This proposition is usually used to prove that a subset of $\mathbb{R}^3$ is <u>not</u> a regular surface.)



Two Propositions

Proposition

Let $p \in S$ *be a point of a regular surface* S *and let* $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ *be a map with* $p \in \mathbf{x}(U) \subset S$ *such that conditions 1 and 3 of the definition hold. Assume that* $\mathbf{x}$ *is one-to-one. Then* $\mathbf{x}^{-1}$ *is continuous.*

The tangent plane

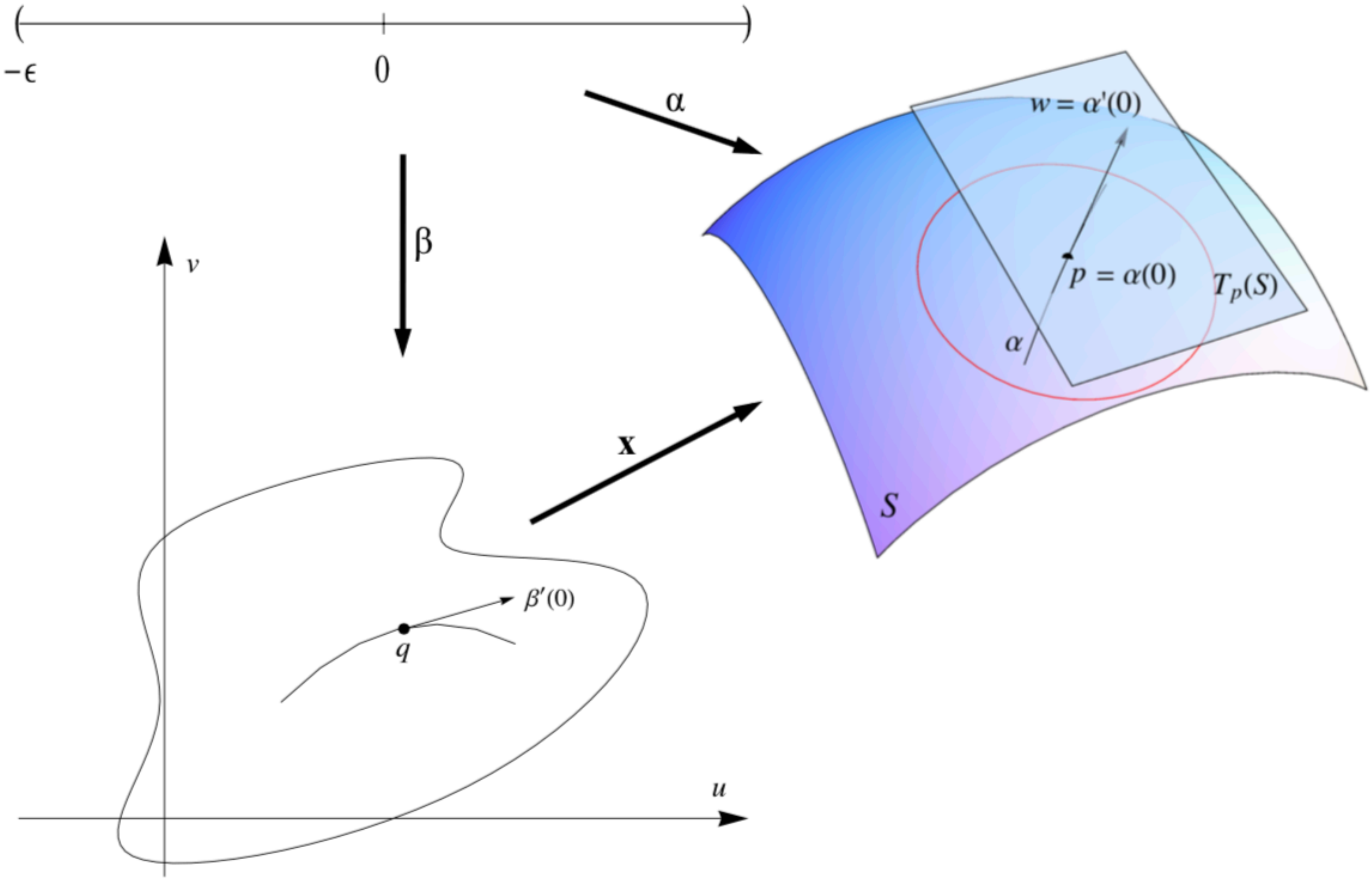
By a *tangent vector* to S, at a point $p \in S$, we mean the tangent vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.

Proposition

Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ *be a parametrization of a regular surface* S *and let* $q \in U$. *The vector subspace of dimension 2,*

$$d\mathbf{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3,$$

coincides with the set of tangent vectors to S *and* $\mathbf{x}(q)$.



$-\epsilon$
0
α
β
v
u
q
$\beta'(0)$
$\mathbf{x}$
$w = \alpha'(0)$
$p = \alpha(0)$
$T_p(S)$
α
S

The Tangent Plane

1. Basis of $T_p(S)$:

2. The coordinate of $w \in T_p(S)$ with respect to $\mathbf{x}_u, \mathbf{x}_v$:

3. Normal Vector $N(p)$ of $T_p(S)$:

By fixing a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ at $p \in S$, we can make a definite choice of a unit normal vector at each point $q \in \mathbf{x}(U)$ by the rule

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_p}{\|\mathbf{x}_u \wedge \mathbf{x}_p\|}(q).$$

Thus, we obtain a differentiable map $N : \mathbf{x}(U) \to \mathbb{R}^3$.

Meaning of "Differentiable" on a curved surface

Definition 6. Let S_1 and S_2 be abstract surfaces. A map $\varphi : S_1 \to S_2$ is *differentiable* at $p \in S_1$ if, given a parametrization $\mathbf{y} : V \subset \mathbb{R}^2 \to S_2$ around $\varphi(p)$, there exists a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S_1$ around p such that $\varphi(\mathbf{x}(U)) \subset \mathbf{y}(V)$ and the map

$$\mathbf{y}^{-1} \circ \varphi \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^2 \tag{1}$$

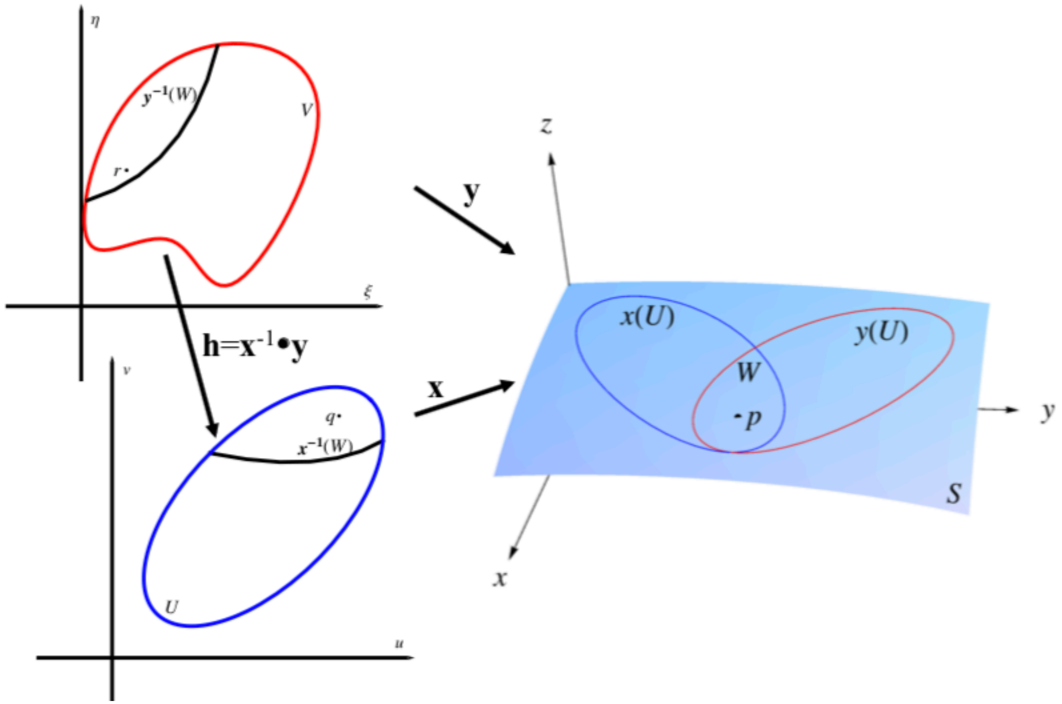
is differentiable at $\mathbf{x}^{-1}(p)$. φ is *differentiable* on S_1 if it is differentiable at every $p \in S_1$.

It is clear, by condition 2, that this definition does not depend on the choices of the parametrizations. The map (1) is called the expression of φ in the parametrizations $\mathbf{x}, \mathbf{y}$.

Change of Parameters

Proposition (∗)

Let p be a point of a regular surface S, and let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, $\mathbf{y} : V \subset \mathbb{R}^2 \to S$ be two parametrizations of S such that $p \in \mathbf{x}(U) \cap \mathbf{y}(V) = W$. Then the "change of coordinates" $h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is a diffeomorphism; that is, h is differentiable and has a differentiable inverse h^{-1}.



KEY:

Thanks to the proposition of "Change of parameters," we can turn this proposition into an axiom in the definition of manifolds.

Defintion for Manifolds:

Definition 5. An *abstract surface* (differentiable manifold of dimension 2) is a set S together with a family of one-to-one maps $\mathbf{x}_\alpha : U_\alpha \to S$ of open sets $U_\alpha \subset \mathbb{R}^2$ into S such that

1. $\bigcup_\alpha \mathbf{x}_\alpha(U_\alpha) = S$.
2. For each pair α, β with $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W \neq \emptyset$, we have that $\mathbf{x}_\alpha^{-1}(W), \mathbf{x}_\beta^{-1}(W)$ are open sets in $\mathbb{R}^2$, and $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$, $\mathbf{x}_\alpha^{-1} \circ \mathbf{x}_\beta$ are differentiable maps.

The pair $(U_\alpha, \mathbf{x}_\alpha)$ with $p \in \mathbf{x}_\alpha(U_\alpha)$ is called a *parametrization* (or coordinate system) of S around p. $\mathbf{x}_\alpha(U_\alpha)$ is called a *coordinate neighborhood*, and if $q = \mathbf{x}_\alpha(u_\alpha, v_\alpha) \in S$, we say that (u_α, v_α) are the *coordinates* of q in this coordinate system. The family $\{U_\alpha, \mathbf{x}_\alpha\}$ is called a *differentiable structure* for S.

It follows immediately from condition 2 that the "change of parameters"

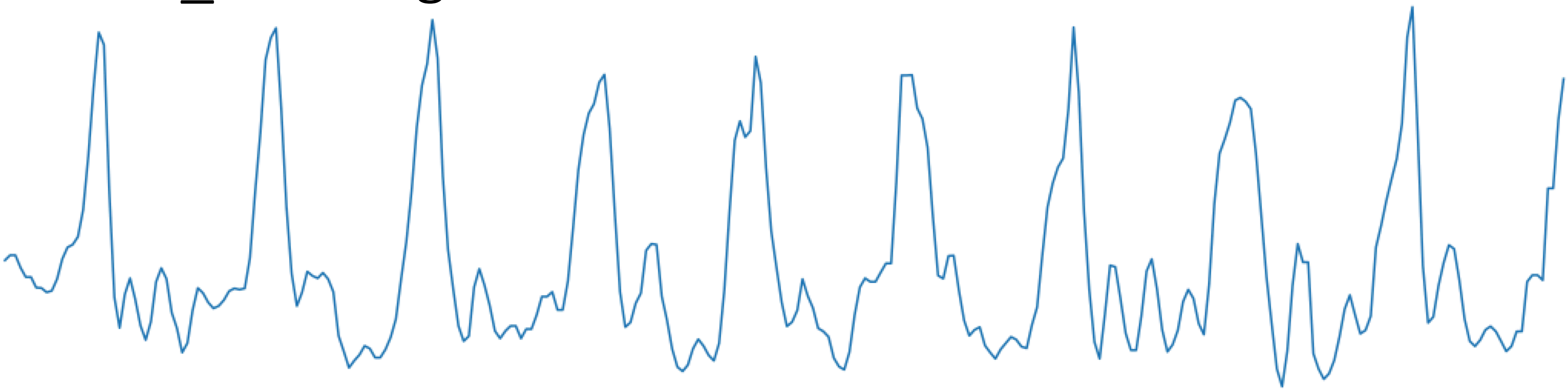
$$\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha : \mathbf{x}_\alpha^{-1}(W) \to \mathbf{x}_\beta^{-1}(W)$$

is a diffeomorphism.

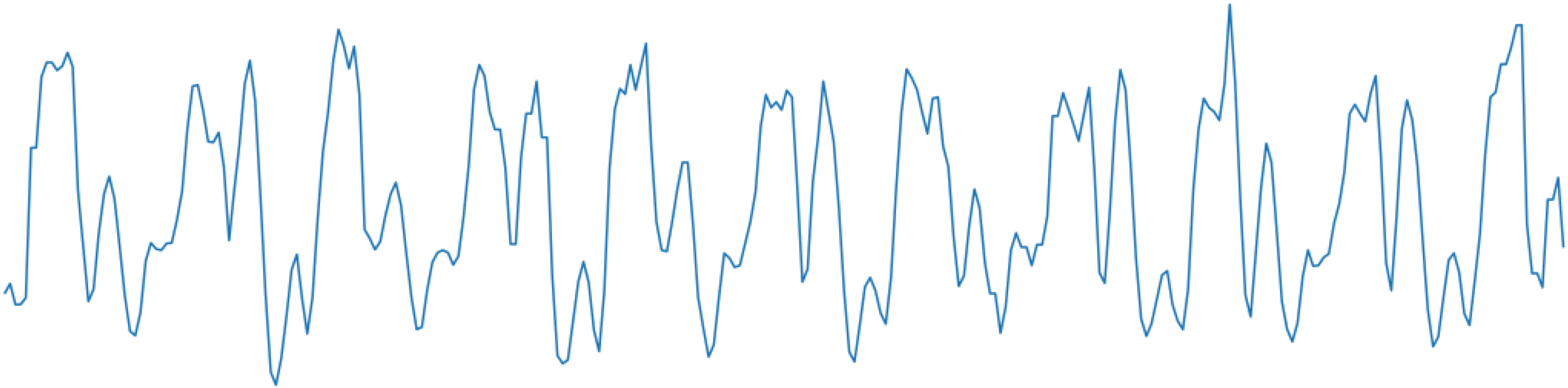
Applications to big data problems

- For cell phone data
- Work out details with students on the board

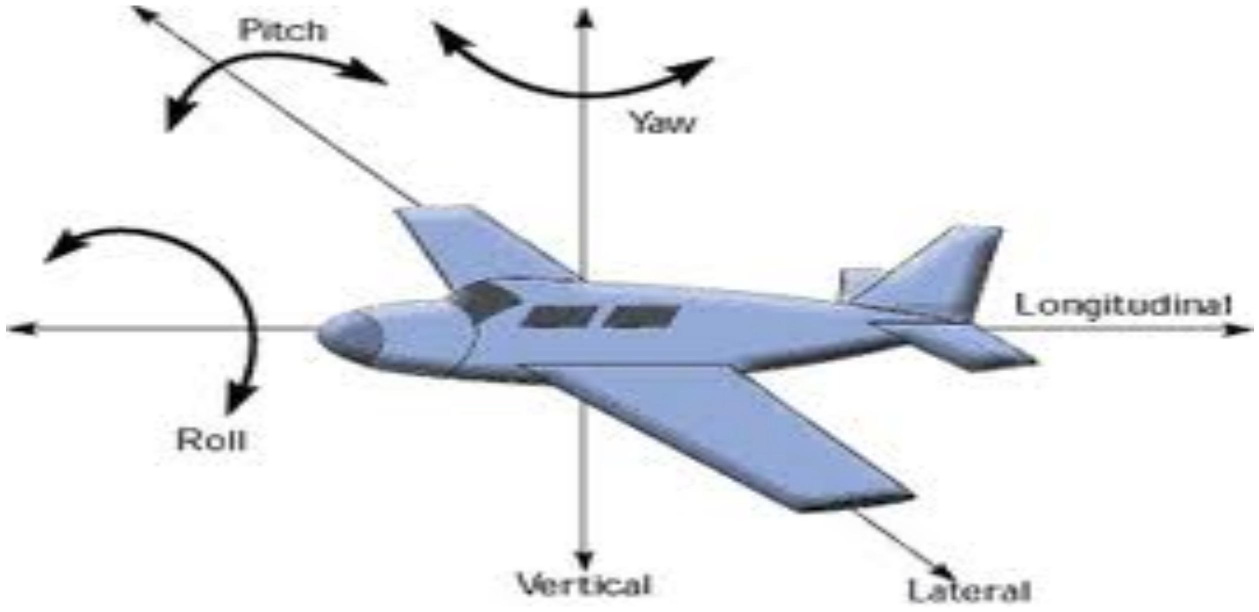
User_i walking data

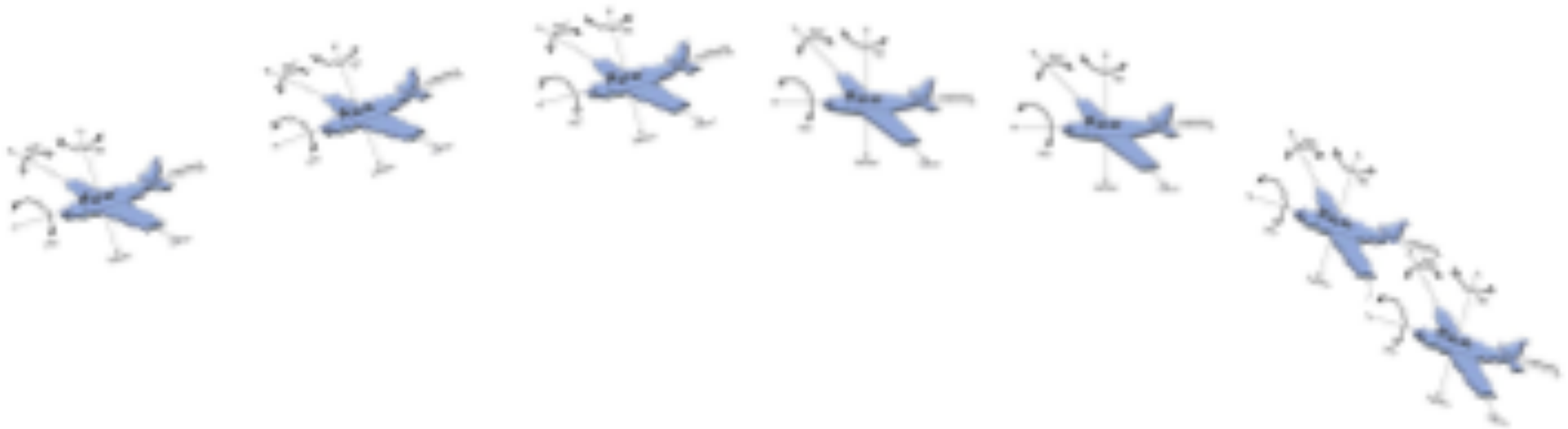


User_j walking data



- **How to model and capture the dynamics and kinematics of an UAV?**



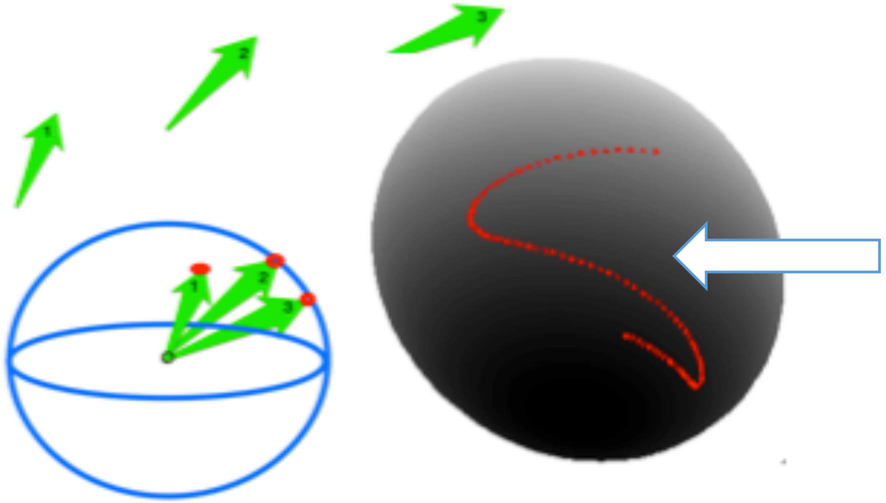


You may wonder: How to use manifold to study UAV data?
Simplest case: drawing a curve on a sphere
Try to capture characteristics of flight controls

- **For example: Only look at UAV "headings"**
- **All possible headings for all UAVs form a sphere.**

Only consider UAV heading directions here, but works for any other UAV characteristics

- ***Key: Developed a dimension-reduction technique for large nonlinear data.***



Just recording the heading while a UAV is flying gives a heading-behavior curve.