Lecture 5

Math 178
Nonlinear Data Analytics

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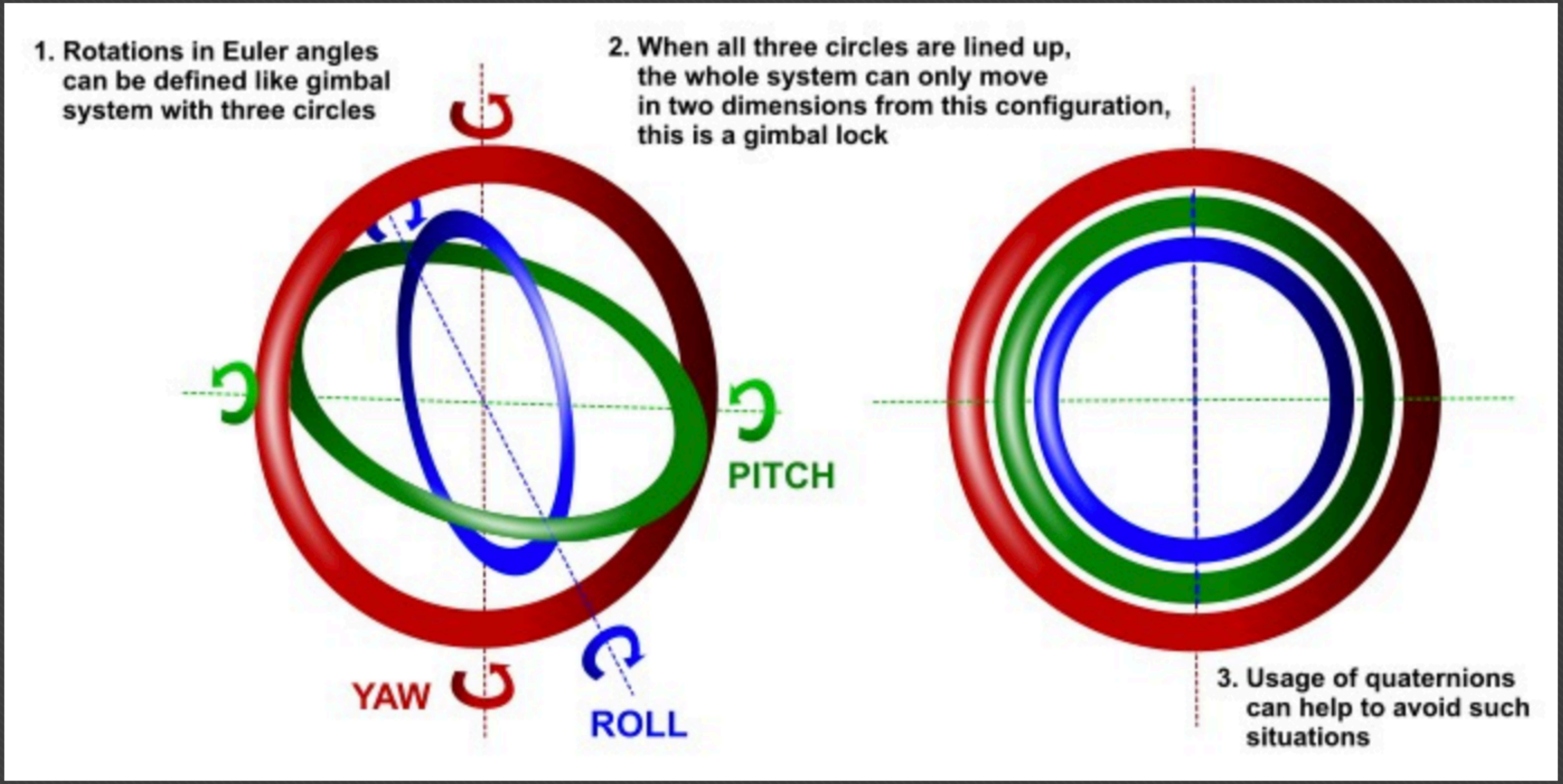
Today

- Cell phone data in quaternion representations
- Gyroscope measurement models
- Accelerometer measurement models
  Choosing the state and modeling its dynamics
- Model for the prior and probabilistic models

Quaternions for Quest3D

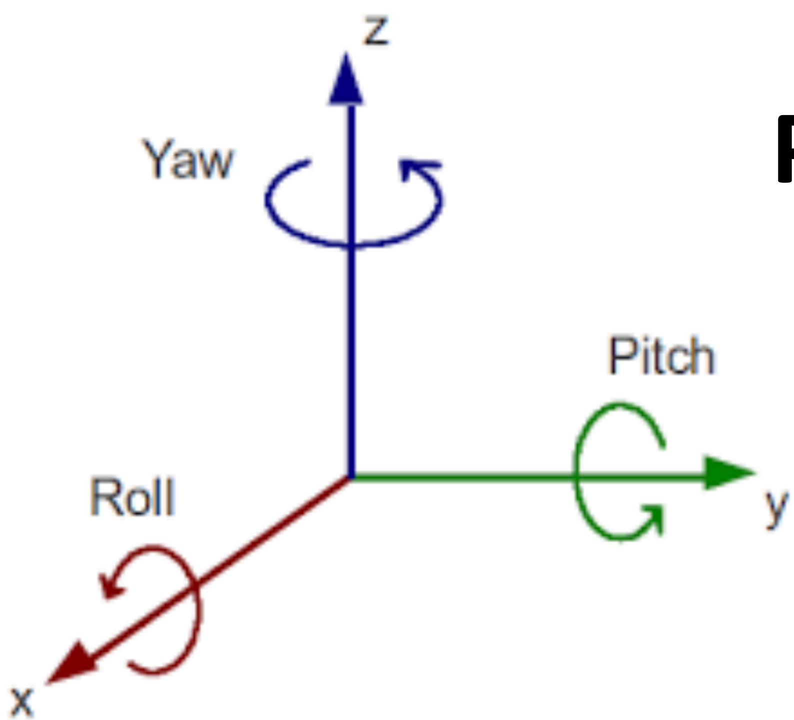
Quaternion is a data type suitable for defining object orientation and rotations. Quaternions are easier to work with than matrices and using quaternions helps to avoid gimbal lock problem like in case of Euler angles usage.

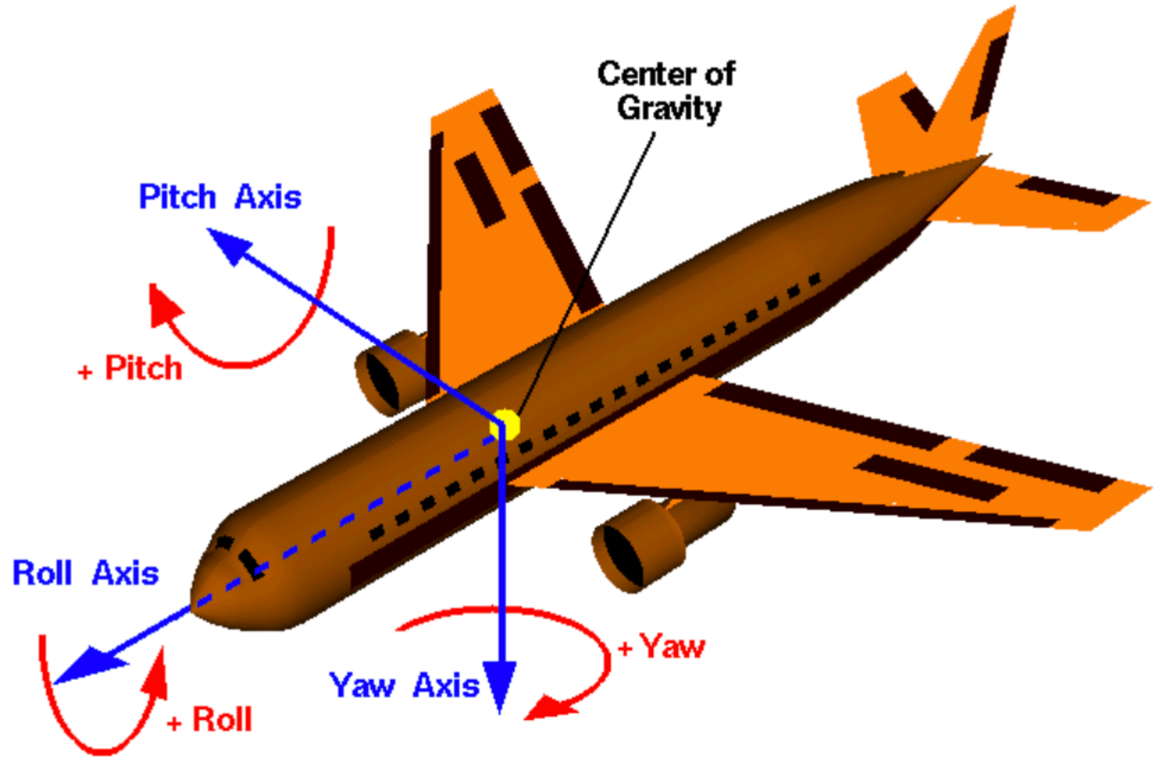
Tasks like smooth interpolation between three-dimensional rotations and building rotation by vector are fairly simpler to solve with quaternions than with Euler angles or matrices. Industrial grade inertial trackers and many other orientation sensors can return rotational data in quaternion form, also to avoid gimbal lock problem, and make such values easier to filter by interpolation.



Gimbal lock problem

Recall:
Pitch, Roll, and Yaw





Why does a unit quaternion represent a rotation and how?

- Work out details with the students on the board.

Unit Quaternion and Euler Angles

- Each unit quaternion can be associated to a rotation around an axis.

$$
\begin{aligned}
\mathbf{q}_0 &= \mathbf{q}_w = \cos(\alpha/2) \\
\mathbf{q}_1 &= \mathbf{q}_x = \sin(\alpha/2)\cos(\beta_x) \\
\mathbf{q}_2 &= \mathbf{q}_y = \sin(\alpha/2)\cos(\beta_y) \\
\mathbf{q}_3 &= \mathbf{q}_z = \sin(\alpha/2)\cos(\beta_z)
\end{aligned}
$$

$$
\mathbf{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T = \begin{bmatrix} q_w & q_x & q_y & q_z \end{bmatrix}^T
$$

$$
|\mathbf{q}|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1
$$

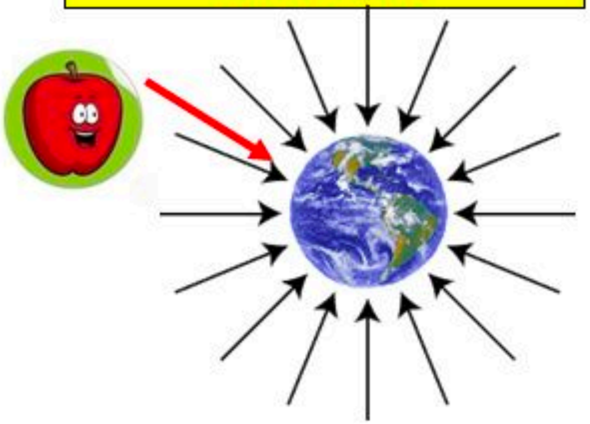
Complication: There are several different Fields, Poles, and Frames

- Gravitational fields
- Electric fields
- Magnetic fields
- Magnetic pole
- Geographic pole
- Heliocentric frame
- Geocentric frame

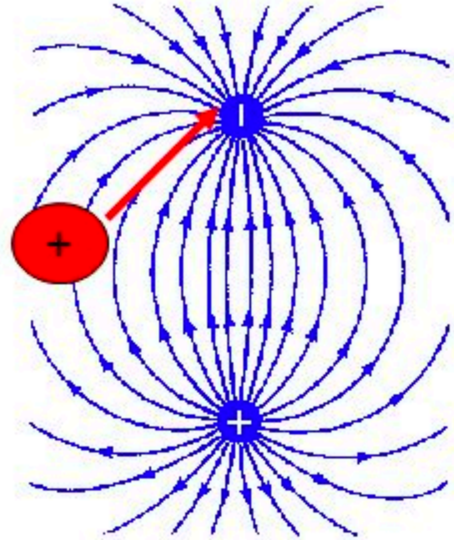
Examples of **gravitational fields**:

- Things falling to Earth
- The Earth orbiting the sun

Test object: Anything with mass



Test object:
charged object (+)



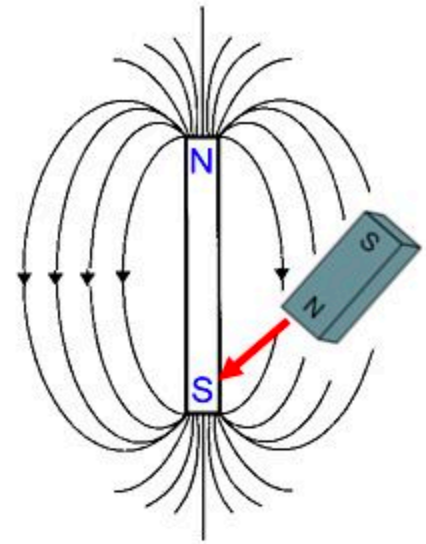
Examples of **electric fields**:

- Static electricity
- Lightning

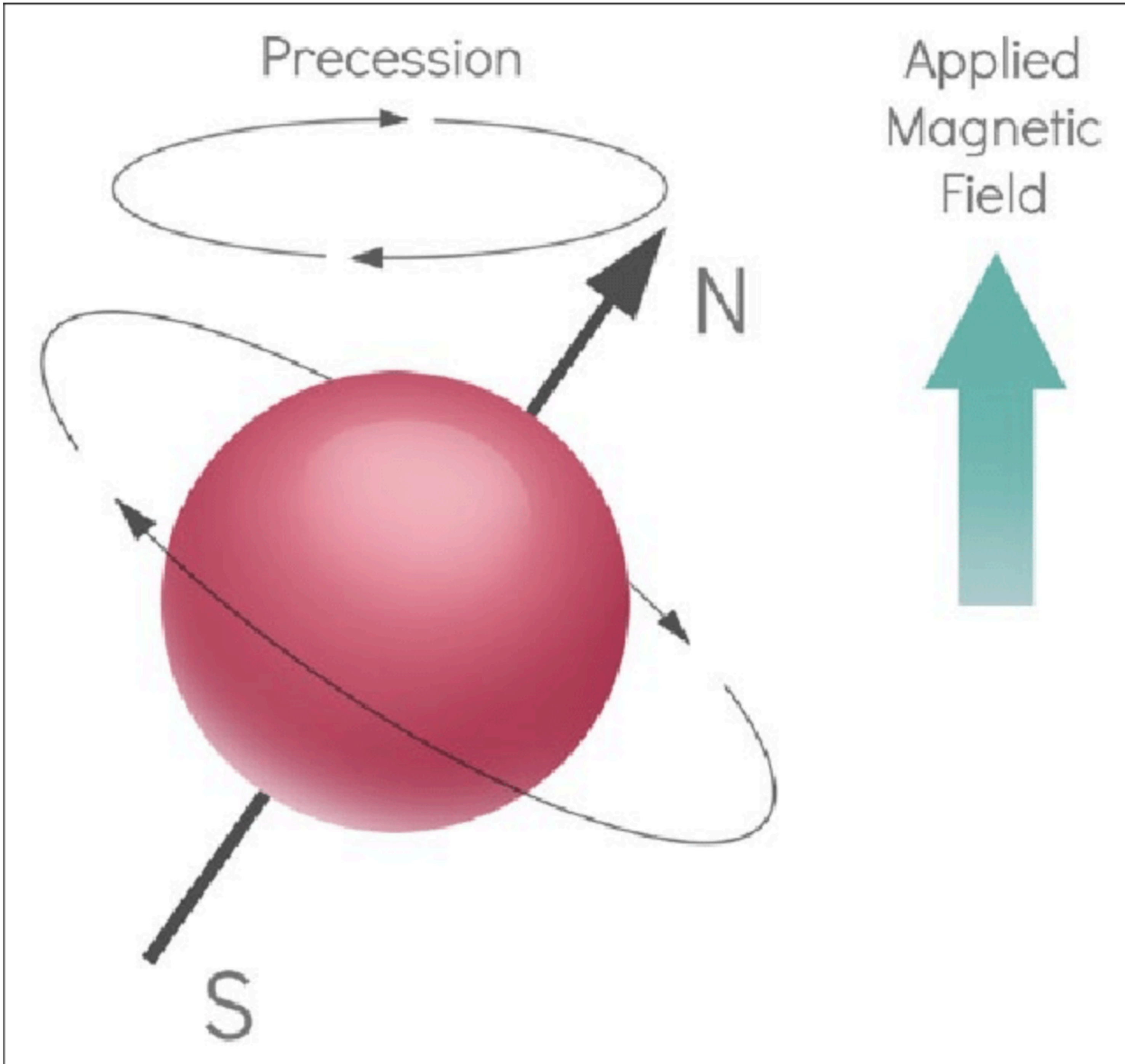
Examples of **magnetic fields**:

- Magnets
- Using a compass

Test object: magnet

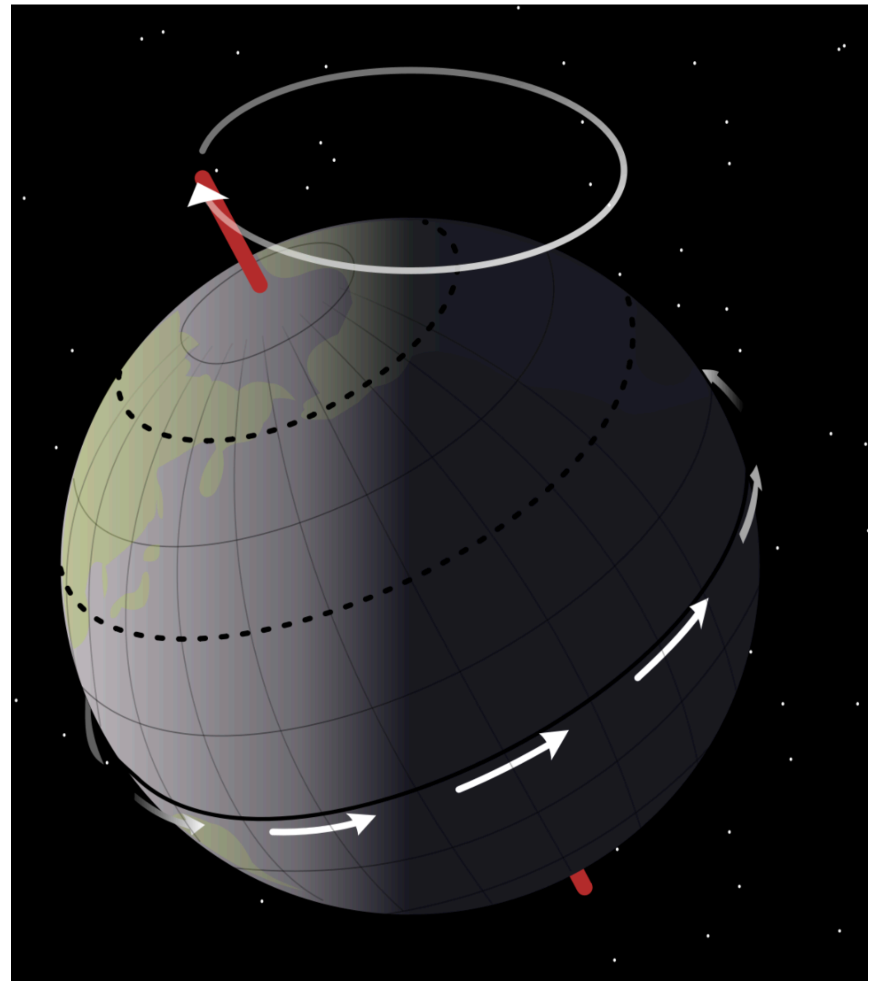


- A field line (or vector diagrams) tells us the direction and strength of a field
  - The direction of a field is determined by the direction a test object will move

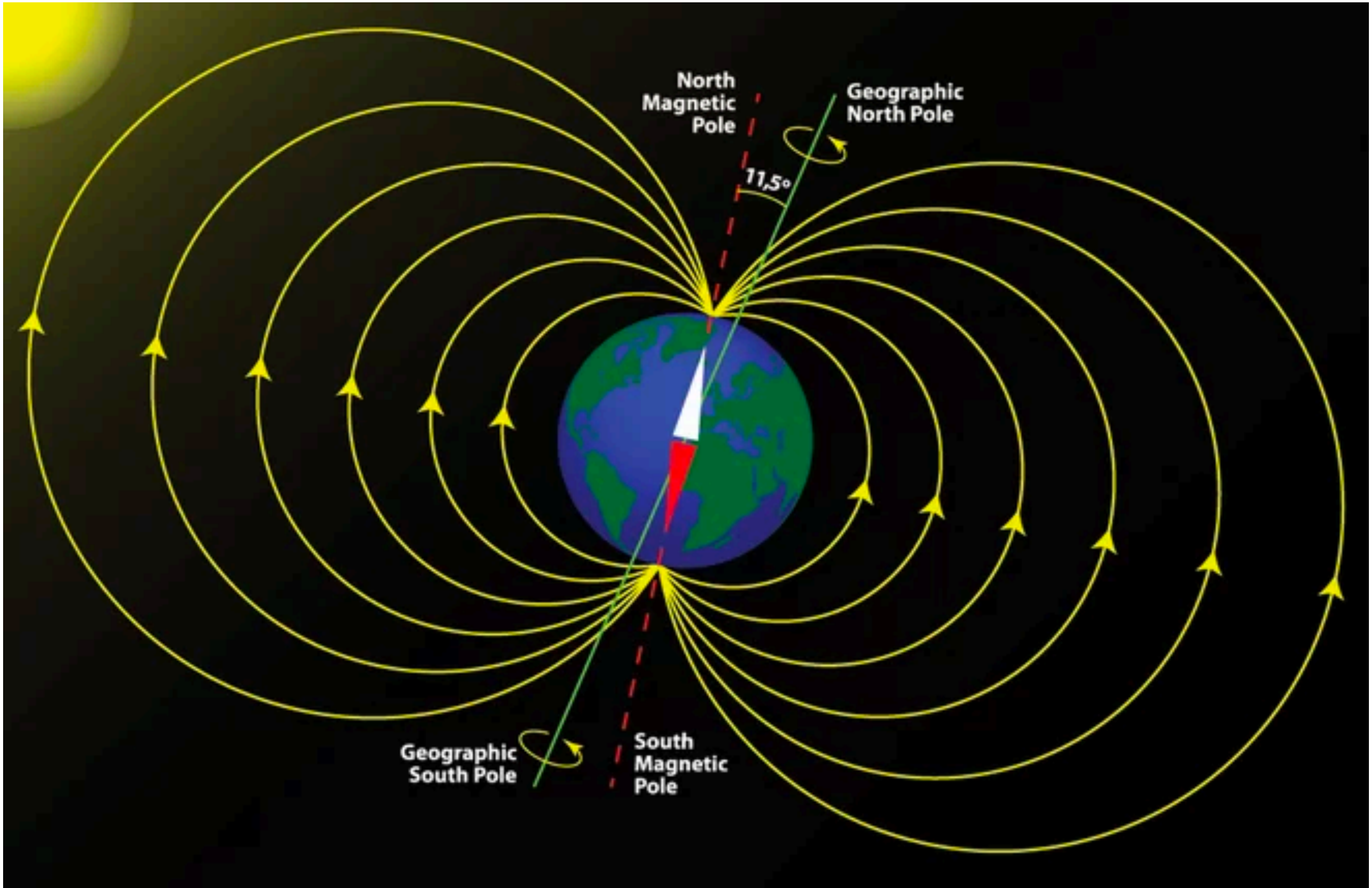

Precession
Applied
Magnetic
Field
N
S

Precession

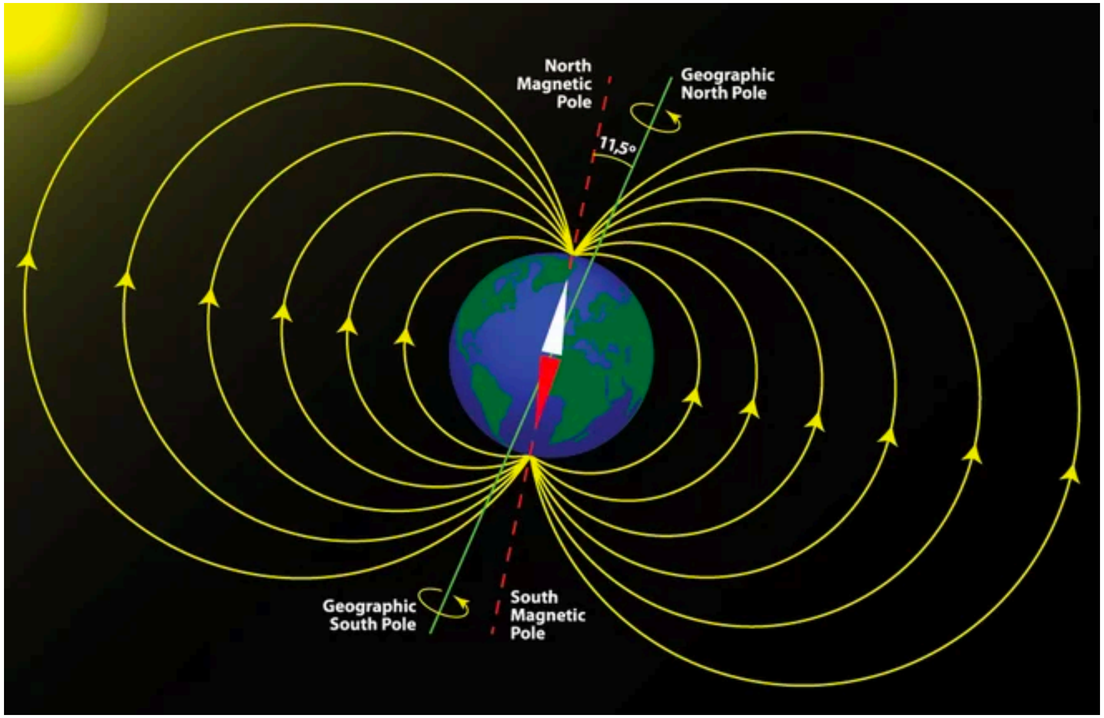
- The slow movement of the axis of a spinning body around another axis due to a torque (such as gravitational influence) acting to change the direction of the first axis. It is seen in the circle slowly traced out by the pole of a spinning gyroscope.



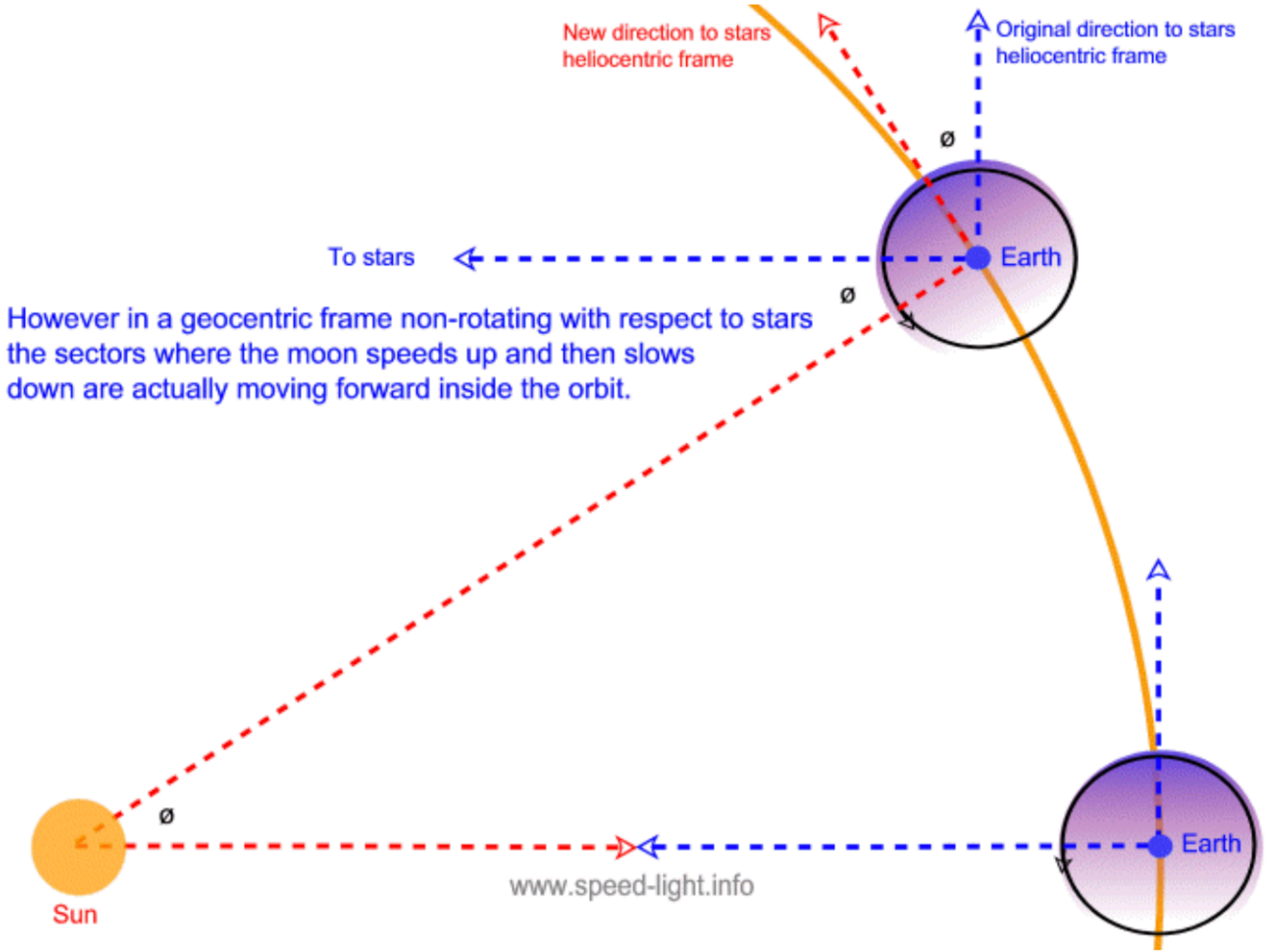
North Magnetic pole and Geographic North pole

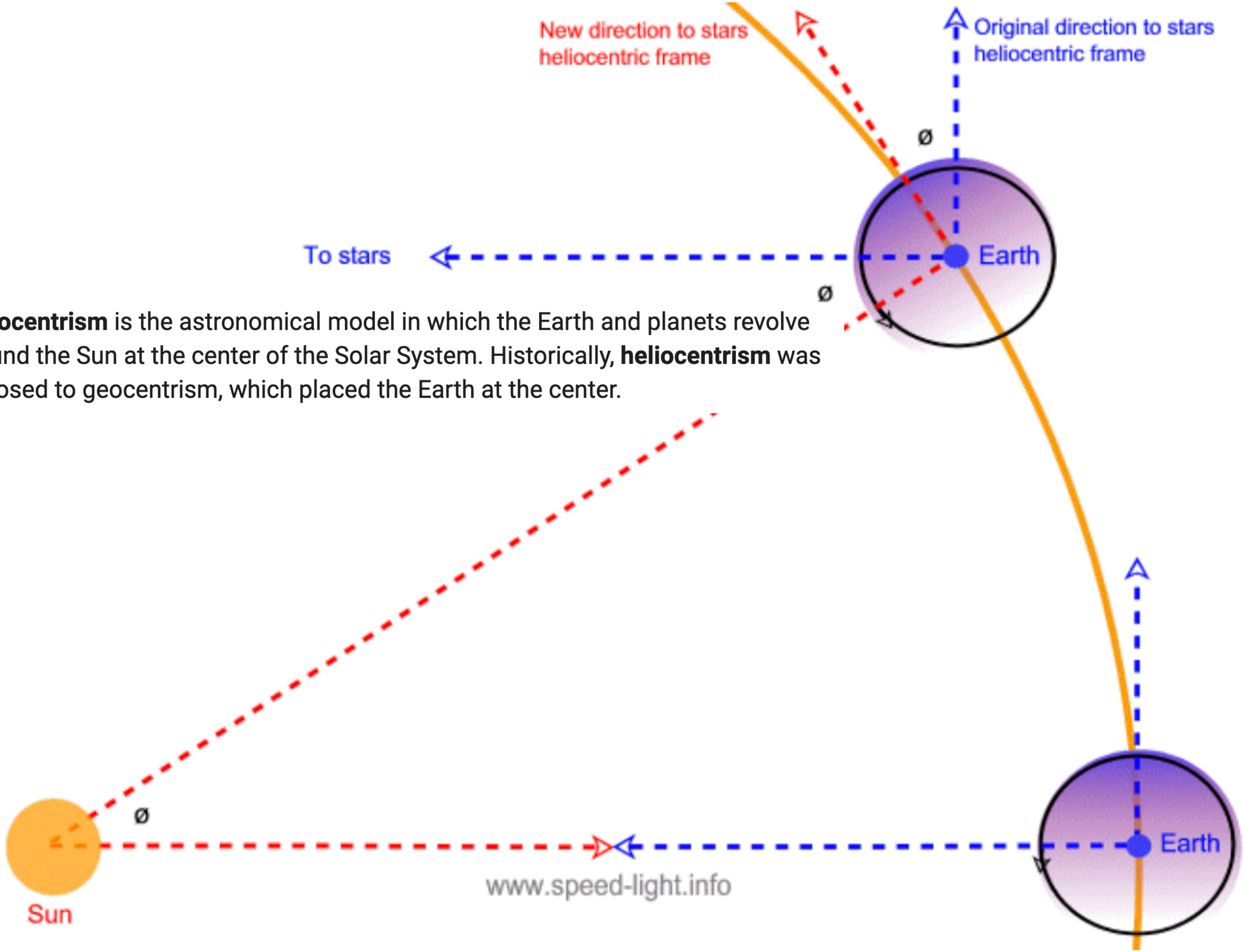


North Magnetic pole
and
Geographic North pole

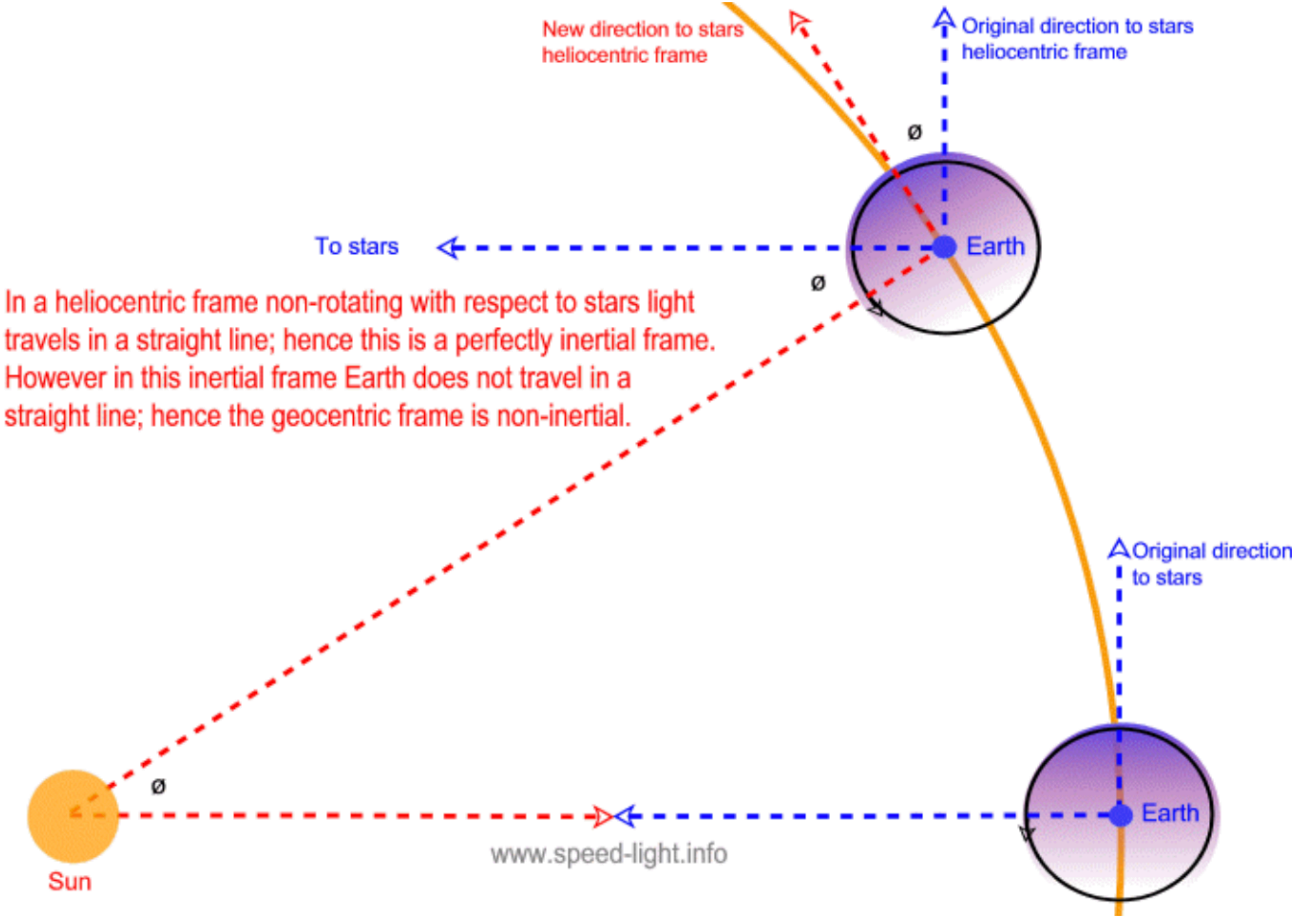


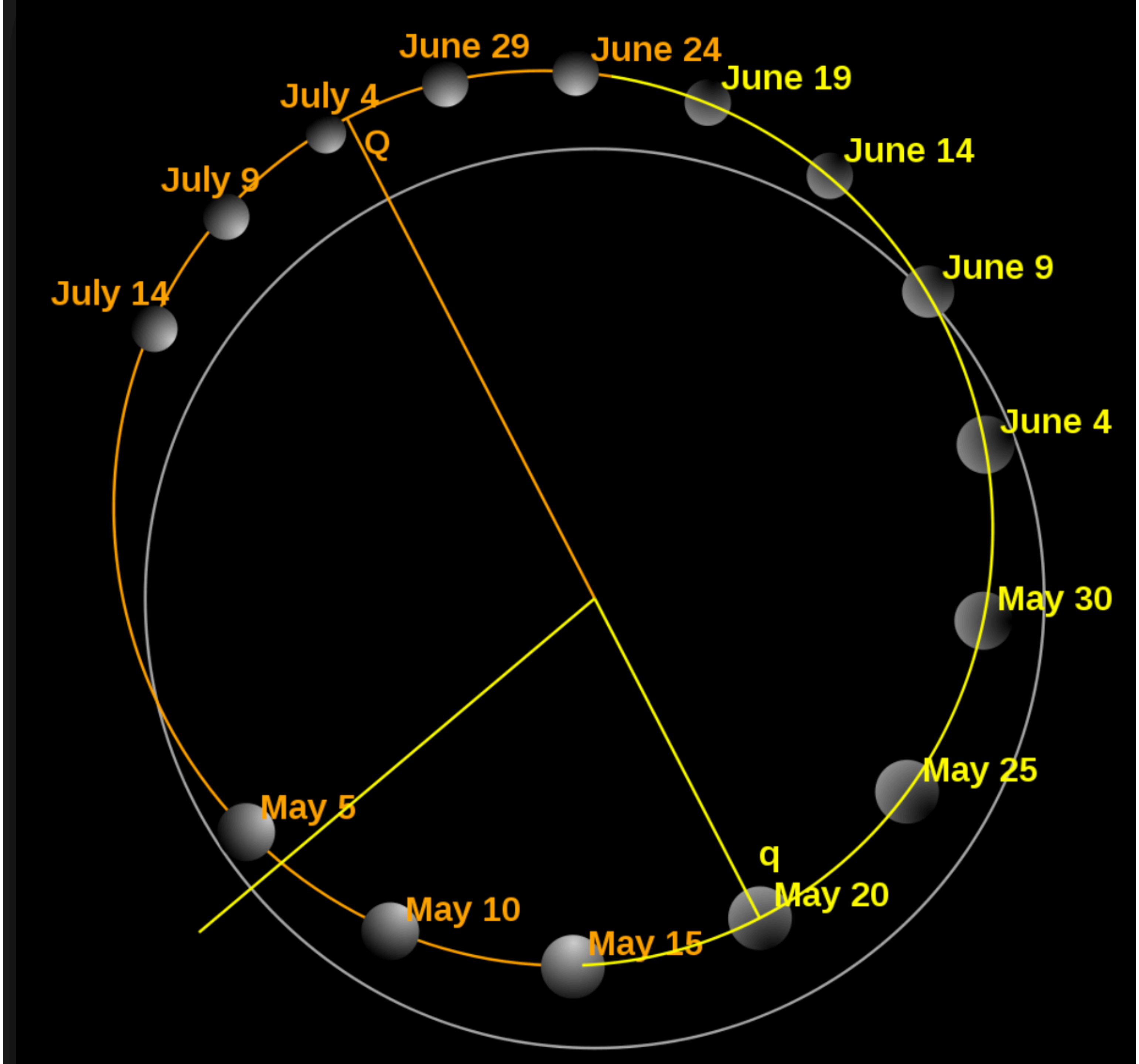
Geographic north and **south poles** are determined by the earth's spin. They are the locations on earth through which the axis **of** the earth's spin passes. **Magnetic north is** determined by the direction a compass points. **Magnetic** variance, or declination, **is** the **difference between geographic north** and **magnetic north**.


New direction to stars
heliocentric frame
Original direction to stars
heliocentric frame
ø
To stars
Earth
ø
However in a geocentric frame non-rotating with respect to stars
the sectors where the moon speeds up and then slows
down are actually moving forward inside the orbit.
ø
Earth
www.speed-light.info
Sun



Heliocentrism is the astronomical model in which the Earth and planets revolve around the Sun at the center of the Solar System. Historically, **heliocentrism** was opposed to geocentrism, which placed the Earth at the center.


New direction to stars
heliocentric frame
Original direction to stars
heliocentric frame
ø
To stars
Earth
ø
In a heliocentric frame non-rotating with respect to stars light travels in a straight line; hence this is a perfectly inertial frame. However in this inertial frame Earth does not travel in a straight line; hence the geocentric frame is non-inertial.
Original direction
to stars
ø
Earth
Sun
www.speed-light.info


June 29
June 24
June 19
July 4
Q
June 14
July 9
June 9
July 14
June 4
May 30
May 25
May 5
q
May 20
May 10
May 15

4 different frames

The body frame $\boldsymbol{b}$ is the coordinate frame of the moving IMU. Its origin is located in the center of the accelerometer triad and it is aligned to the casing. All the inertial measurements are resolved in this frame.

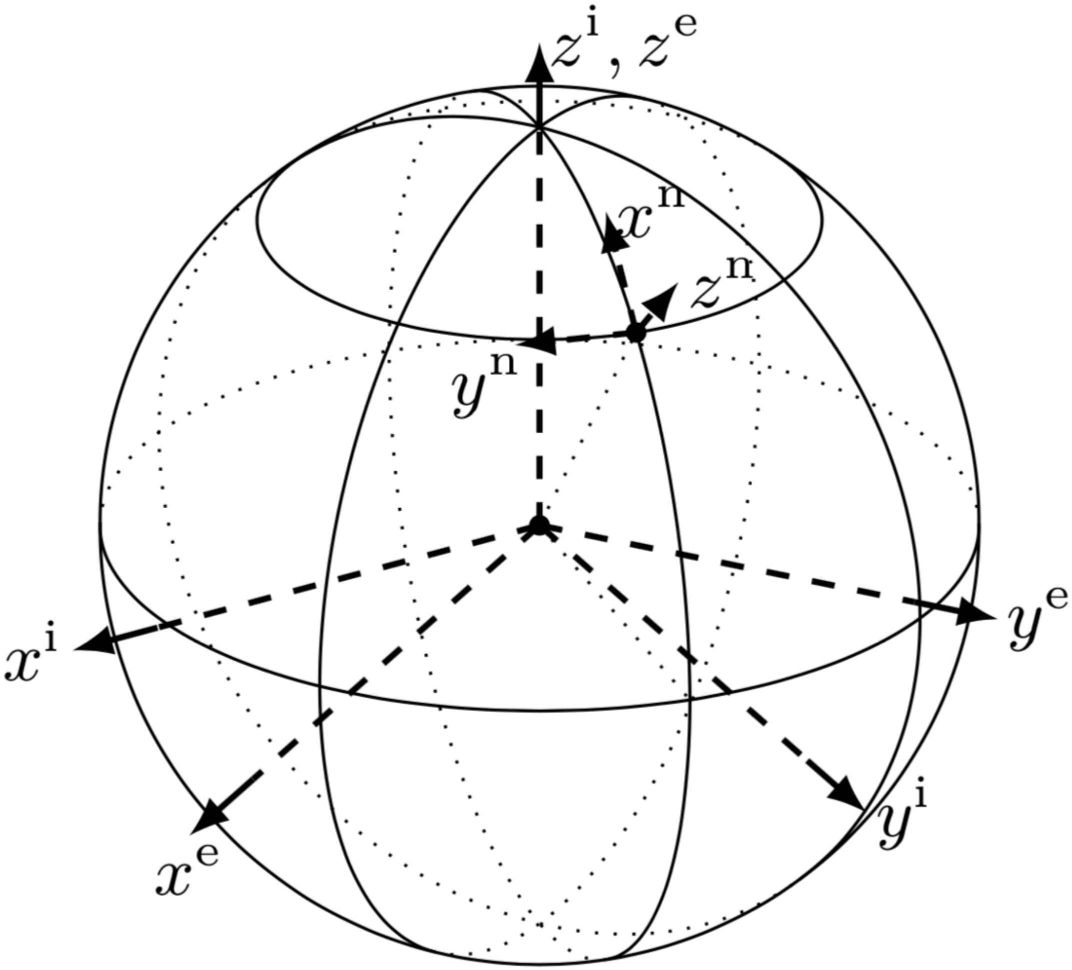
The navigation frame $\boldsymbol{n}$ is a local geographic frame in which we want to navigate. In other words, we are interested in the position and orientation of the b-frame with respect to this frame. For most applications it is defined stationary with respect to the earth. However, in cases when the sensor is expected to move over large distances, it is customary to move and rotate the n-frame along the surface of the earth. The first definition is used throughout this tutorial, unless mentioned explicitly.

The inertial frame $\boldsymbol{i}$ is a stationary frame. The IMU measures linear acceleration and angular velocity with respect to this frame. Its origin is located at the center of the earth and its axes are aligned with respect to the stars.

The earth frame $\boldsymbol{e}$ coincides with the i-frame, but rotates with the earth. That is, it has its origin at the center of the earth and axes which are fixed with respect to the earth.

Using subscripts b, e, n, I to denote the four different frames

- the n-frame at a certain location on the earth,
- the e-frame rotating with the earth and
- the i-frame.



- A **gyroscope measures** the sensor’s angular velocity, i.e. the rate of change of the sensor’s orientation.
- An **accelerometer measures** the external specific force acting on the sensor.

Inertial Sensor and IMUs

- The term **inertial sensor** is used to denote the combination of a three-axis accelerometer and a three- axis gyroscope.
- Devices containing these sensors are commonly referred to as inertial measurement units (**IMUs**).
- Inertial sensors are nowadays also present in most modern smartphone, and in devices such as Wii controllers and virtual reality (VR) headsets.

(a) Left bottom: an Xsens MTx IMU [155]. Left top: a Trivisio Colibri Wireless IMU [147]. Right: a Samsung Galaxy S4 mini smartphone.

(b) A Samsung gear VR.[1]

(c) A Wii controller containing an accelerometer and a MotionPlus expansion device containing a gyroscope.[2]

MEMS has large # of Applications. Bellow all use of a single IMU placed on a moving object to estimate its pose.

(a) Back pain therapy using serious gaming. IMUs are placed on the chest-bone and on the pelvis to estimate the movement of the upper body and pelvis. This movement is used to control a robot in the game and promotes movements to reduce back pain.

(b) Actor Seth MacFarlane wearing 17 IMUs to capture his motion and animate the bear Ted. The IMUs are placed on different body segments and provide information about the relative position and orientation of each of these segments.

(a) Inertial sensors are used in combination with GNSS measurements to estimate the position of the cars in a challenge on cooperative and autonomous driving.

(b) Due to their small size and low weight, IMUs can be used to estimate the orientation for control of an unmanned helicopter.

Dead-reckoning

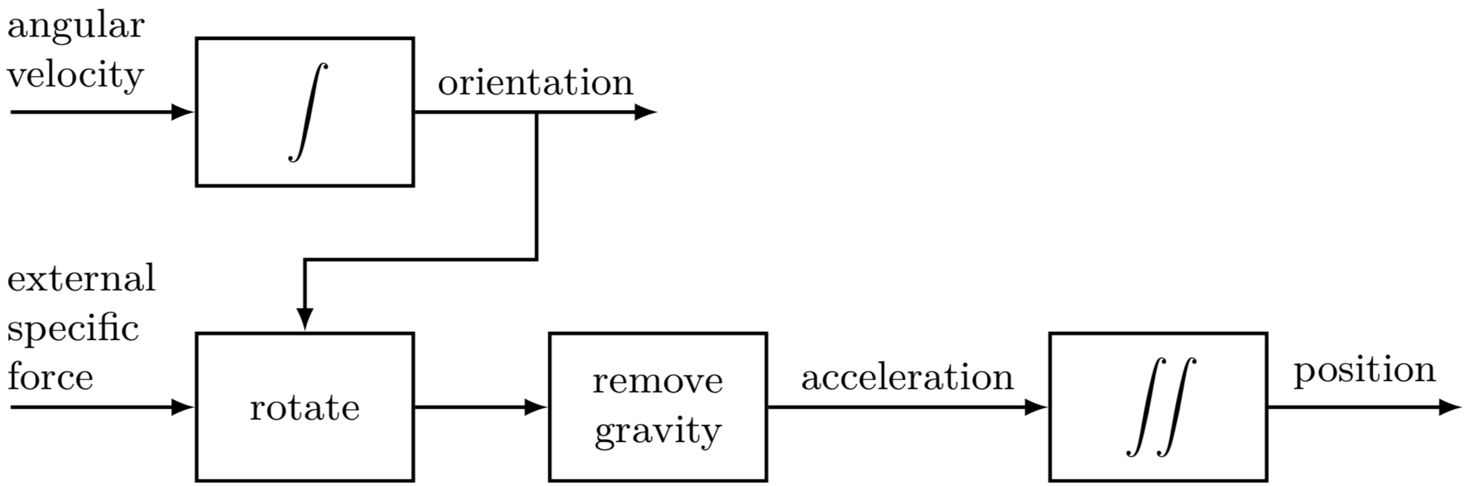


Figure 1.4: Schematic illustration of dead-reckoning, where the accelerometer measurements (external specific force) and the gyroscope measurements (angular velocity) are integrated to position and orientation.

Caution!

- Accelerometer data is not the record of acceleration!

we must understand rotations even when we just want to use accelerometer data in our analysis

- Why? Say if we want to get the position data of the user, we can not just simply integrate twice of accelerometer data since accelerometer data is not acceleration (the second derivative of the position);
- rather the accelerometer data is recorded by an accelerometer which measures some specific force f in the body frame.

The relation between this force f (the accelerometer data) and the linear acceleration a is given below:

$$f^{\mathrm{b}} = R^{\mathrm{bn}}(a^{\mathrm{n}}_{\mathrm{ii}} - g^{\mathrm{n}}),$$

where

R denote the rotation,

g denotes the gravity vector and

a denotes the linear acceleration of the sensor expressed in navigation frame.

We use a superscript to indicate in which coordinate frame a vector is expressed.

Again: Four different frames
we are going to use

- 1. The cell phone body frame **b**.
- 2. The navigation frame **n**.
- 3. The inertial frame **i**. *(This is the fixed frame! Note it does not even depending on the earth's rotation. We have to do analysis by moving everything into this frame.)*
- 4. The earth frame **e**.

4 different frames

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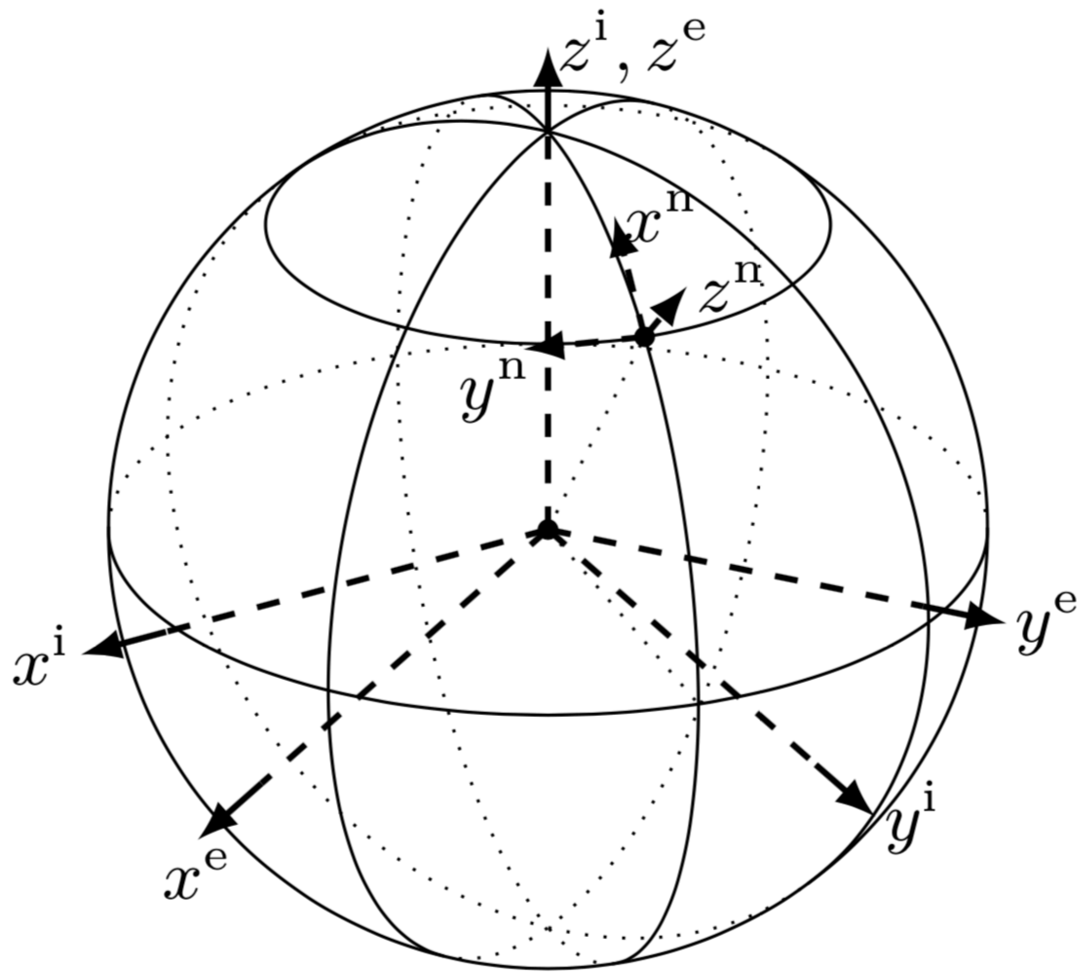
- the n-frame at a certain location on the earth,
- the e-frame rotating with the earth and
- the i-frame.



How to rotate vectors from one frame to another?

Example 2.1 (Rotation of vectors to different coordinate frames) *Consider a vector x expressed in the body frame b. We denote this vector x^b. The rotation matrix R^{nb} rotates a vector from the body frame b to the navigation frame n. Conversely, the rotation from navigation frame n to body frame b is denoted $R^{bn} = (R^{nb})^\mathsf{T}$. Hence, the vector x expressed in the body frame (x^b) and expressed in the navigation frame (x^n) are related according to*

$$x^n = R^{nb} x^b, \qquad x^b = (R^{nb})^\mathsf{T} x^n = R^{bn} x^n. \tag{2.1}$$

R^{nb}

Rotate from from b to n

$X^n = R^{nb} X^b$

As if b cancelled, left n

What does a gyroscope exactly measure?

The gyroscope measures the angular velocity of the (cell phone) body frame with respect to the inertial frame, expressed in the body frame,

denoted by $\omega_{\mathrm{ib}}^{\mathrm{b}}$. This angular velocity can be expressed as

$$\omega_{\mathrm{ib}}^{\mathrm{b}} = R^{\mathrm{bn}} \left(\omega_{\mathrm{ie}}^{\mathrm{n}} + \omega_{\mathrm{en}}^{\mathrm{n}} \right) + \omega_{\mathrm{nb}}^{\mathrm{b}},$$

where R^{bn} is the rotation matrix from the navigation frame to the body frame. The *earth rate*, *i.e.* the angular velocity of the earth frame with respect to the inertial frame is denoted by ω_{ie}. The earth rotates around its own z-axis in 23.9345 hours with respect to the stars [101]. Hence, the earth rate is approximately $7.29 \cdot 10^{-5}$ rad/s.

In case the navigation frame is not defined stationary with respect to the earth, the angular velocity ω_{en}, *i.e.* the *transport rate* is non-zero. The angular velocity required for navigation purposes — in which we are interested when determining the orientation of the body frame with respect to the navigation frame — is denoted by ω_{nb}.

What does a accelerometer exactly measure?

- **The accelerometer measures the specific force f in the body frame b.** This can be expressed as

$$f^{\mathrm{b}} = R^{\mathrm{bn}}(a^{\mathrm{n}}_{\mathrm{ii}} - g^{\mathrm{n}}),$$

where g denotes the gravity vector and an $a^{\mathrm{n}}_{\mathrm{ii}}$ denotes the linear acceleration of the sensor expressed in the navigation frame, which is

$$a^{\mathrm{n}}_{\mathrm{ii}} = R^{\mathrm{ne}} R^{\mathrm{ei}} a^{\mathrm{i}}_{\mathrm{ii}}.$$

The subscripts are used to indicate in which frame the differentiation is performed.

Ask yourself: In which frame the derivative was taken?

For example:

For navigation purposes, we are interested in the position of the sensor in the navigation frame p^n and its derivatives as performed in the navigation frame:

$$\frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{n}} \Big|_{\mathrm{n}} = v^{\mathrm{n}}_{\mathrm{n}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} v^{\mathrm{n}} \Big|_{\mathrm{n}} = a^{\mathrm{n}}_{\mathrm{nn}}.$$

How are a_{ii} and a_{nn} are exactly related?

- A relation between a_{ii} and a_{nn} can be derived by using the relation between two rotating coordinate frames. Given a vector x in a coordinate frame u,

Like a product rule, but be caution

$$\frac{\mathrm{d}}{\mathrm{d}t} x^{\mathrm{u}}\Big|_{\mathrm{u}} = \frac{\mathrm{d}}{\mathrm{d}t} R^{\mathrm{uv}} x^{\mathrm{v}}\Big|_{\mathrm{u}} = R^{\mathrm{uv}} \frac{\mathrm{d}}{\mathrm{d}t} x^{\mathrm{v}}\Big|_{\mathrm{v}} + \omega_{\mathrm{uv}}^{\mathrm{u}} \times x^{\mathrm{u}},$$

where $\omega_{\mathrm{uv}}^{\mathrm{u}}$ is the angular velocity of the v-frame with respect to the u-frame, expressed in the u-frame.

where we have use the two equations on previous 2 slides and use the fact that the angular velocity of the earth is constant,

$$i.e.\ \frac{\mathrm{d}}{\mathrm{d}t} \omega_{\mathrm{ie}}^{\mathrm{i}} = 0.$$

We often want to view v_i and a_{ii} in the inertial frame. How?

- Using the fact that

$$p^{\mathrm{i}} = R^{\mathrm{ie}} p^{\mathrm{e}},$$

the velocity v_{i} and acceleration a_{ii} can be expressed as

$$v_{\mathrm{i}}^{\mathrm{i}} = \frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{i}}\Big|_{\mathrm{i}} = \frac{\mathrm{d}}{\mathrm{d}t} R^{\mathrm{ie}} p^{\mathrm{e}}\Big|_{\mathrm{i}} = R^{\mathrm{ie}} \frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{e}}\Big|_{\mathrm{e}} + \omega_{\mathrm{ie}}^{\mathrm{i}} \times p^{\mathrm{i}} = v_{\mathrm{e}}^{\mathrm{i}} + \omega_{\mathrm{ie}}^{\mathrm{i}} \times p^{\mathrm{i}}, \tag{2.8a}$$

$$a_{\mathrm{ii}}^{\mathrm{i}} = \frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{i}}^{\mathrm{i}}\Big|_{\mathrm{i}} = \frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{e}}^{\mathrm{i}}\Big|_{\mathrm{i}} + \frac{\mathrm{d}}{\mathrm{d}t} \omega_{\mathrm{ie}}^{\mathrm{i}} \times p^{\mathrm{i}}\Big|_{\mathrm{i}} = a_{\mathrm{ee}}^{\mathrm{i}} + 2\omega_{\mathrm{ie}}^{\mathrm{i}} \times v_{\mathrm{e}}^{\mathrm{i}} + \omega_{\mathrm{ie}}^{\mathrm{i}} \times \omega_{\mathrm{ie}}^{\mathrm{i}} \times p^{\mathrm{i}}, \tag{2.8b}$$

Similarly we can express velocity v and acceleration a in earth coordinates

Using the relation between the earth and navigation frames,

$$p^{\mathrm{e}} = R^{\mathrm{en}} p^{\mathrm{n}} + n^{\mathrm{e}}_{\mathrm{ne}},$$

where n_{ne} is the distance from the origin of the earth coordinate frame to the origin of the navigation coordinate frame, expressions similar to (2.8) can be derived. Note that in general it can not be assumed that $\frac{\mathrm{d}}{\mathrm{d}t}\omega_{\mathrm{en}} = 0$. Inserting the obtained expressions into (2.8), it is possible to derive the relation between a_{ii} and a_{nn}. Instead of deriving these relations, we will assume that the navigation frame is fixed to the earth frame, and hence R^{en} and $n^{\mathrm{e}}_{\mathrm{ne}}$ are constant and

$$v^{\mathrm{e}}_{\mathrm{e}} = \frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{e}}\Big|_{\mathrm{e}} = \frac{\mathrm{d}}{\mathrm{d}t} R^{\mathrm{en}} p^{\mathrm{n}}\Big|_{\mathrm{e}} = R^{\mathrm{en}} \ \frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{n}}\Big|_{\mathrm{n}} = v^{\mathrm{e}}_{\mathrm{n}}, \tag{2.10a}$$

$$a^{\mathrm{e}}_{\mathrm{ee}} = \frac{\mathrm{d}}{\mathrm{d}t} v^{\mathrm{e}}_{\mathrm{e}}\Big|_{\mathrm{e}} = \frac{\mathrm{d}}{\mathrm{d}t} v^{\mathrm{e}}_{\mathrm{n}}\Big|_{\mathrm{n}} = a^{\mathrm{e}}_{\mathrm{nn}}. \tag{2.10b}$$

- This is a reasonable assumption as long as the sensor does not travel over significant distances as compared to the size of the earth and it will be one of the model assumptions that we will use in this course.

Now we can derive the relation of accelerations in different frames.

Inserting (2.10) into (2.8) and rotating the result, it is possible to express a^n_{ii} in terms of a^n_{nn} as

$$a^n_{ii} = a^n_{nn} + 2\omega^n_{ie} \times v^n_n + \omega^n_{ie} \times \omega^n_{ie} \times p^n,$$

Coriolis acceleration *centrifugal acceleration*

This is a reasonable assumption as long as the sensor does not travel over significant distances as compared to the size of the earth and it will be one of the model assumptions.

Because this model assumption, it is not a good idea to put all the data collected from everywhere in this world into on data set. Keeping them apart has its advantages.

Example 2.2 (Magnitude of centrifugal and Coriolis acceleration) *The centrifugal acceleration depends on the location on the earth. It is possible to get a feeling for its magnitude by considering the property of the cross product stating that*

$$\|\omega_{ie}^n \times \omega_{ie}^n \times p^n\|_2 \leq \|\omega_{ie}^n\|_2 \|\omega_{ie}^n\|_2 \|p^n\|_2. \tag{2.12}$$

Since the magnitude of ω_{ie} *is approximately* $7.29 \cdot 10^{-5}$ rad/s *and the average radius of the earth is* 6371 km *[101], the magnitude of the centrifugal acceleration is less than or equal to* $3.39 \cdot 10^{-2}$ m/s^2.

The Coriolis acceleration depends on the speed of the sensor. Let us consider a person walking at a speed of 5 km/h. *In that case the magnitude of the Coriolis acceleration is approximately* $2.03 \cdot 10^{-4}$ m/s^2. *For a car traveling at* 120 km/h, *the magnitude of the Coriolis acceleration is instead* $4.86 \cdot 10^{-3}$ m/s^2.

We can use them to detect whether a person is in car or not.

Get used to see things from different points of view, especially for rotations

Example 3.2 (Rotation of a coordinate frame and rotation of a vector) *Consider the 2D example in Figure 3.3, where on the left, a vector* x *is rotated clockwise by an angle* α *to* $x_\star$*. This is equivalent to (on the right) rotating the coordinate frame* v *counterclockwise by an angle* α*. Note that* $x_\star^v = x^u$.

Figure 3.3: Left: clockwise rotation α of the vector x to the vector $x_\star$. Right: counterclockwise rotation α of the coordinate frame v to the coordinate frame u.

In Figure 3.4, a vector x is rotated an angle α around the unit vector n. We denote the rotated vector by $x_\star$. Suppose that x as expressed in the coordinate frame v is known (and denoted x^{v}) and that we want to express $x_\star^{\text{v}}$ in terms of x^{v}, α and n.

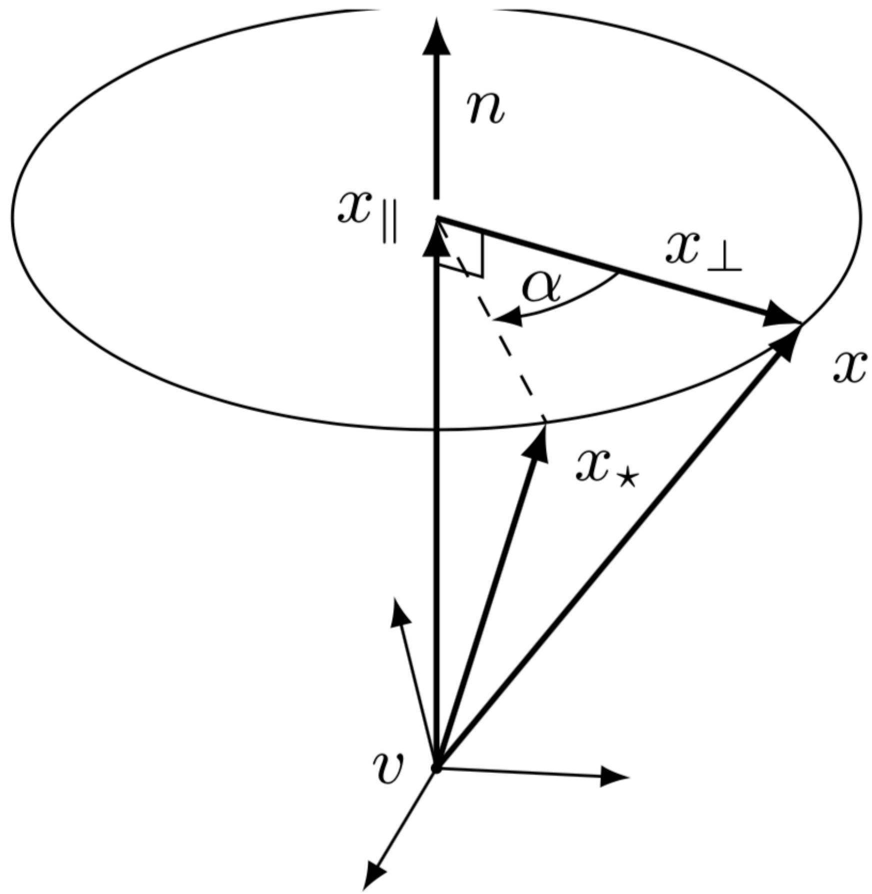


Figure 3.4: Clockwise rotation of a vector x by an angle α around the unit vector n. The rotated vector is denoted by $x_\star$. The vector x is decomposed in a component $x_\|$ that is parallel to the axis n, and a component $x_\perp$ that is orthogonal to it.

It can first be recognized that the vector x can be decomposed into a component parallel to the axis n, denoted $x_{\|}$, and a component orthogonal to it, denoted $x_{\perp}$, as

$$x^{\mathrm{v}} = x_{\|}^{\mathrm{v}} + x_{\perp}^{\mathrm{v}}. \tag{3.10a}$$

Based on geometric reasoning we can conclude that

$$x_{\|}^{\mathrm{v}} = (x^{\mathrm{v}} \cdot n^{\mathrm{v}})\, n^{\mathrm{v}}, \tag{3.10b}$$

where $\cdot$ denotes the inner product. Similarly, $x_{\star}^{\mathrm{v}}$ can be decomposed as

$$x_{\star}^{\mathrm{v}} = (x_{\star}^{\mathrm{v}})_{\|} + (x_{\star}^{\mathrm{v}})_{\perp}, \tag{3.11a}$$

where

$$(x_{\star}^{\mathrm{v}})_{\|} = x_{\|}^{\mathrm{v}}, \tag{3.11b}$$

$$(x_{\star}^{\mathrm{v}})_{\perp} = x_{\perp}^{\mathrm{v}} \cos\alpha + (x^{\mathrm{v}} \times n^{\mathrm{v}}) \sin\alpha. \tag{3.11c}$$

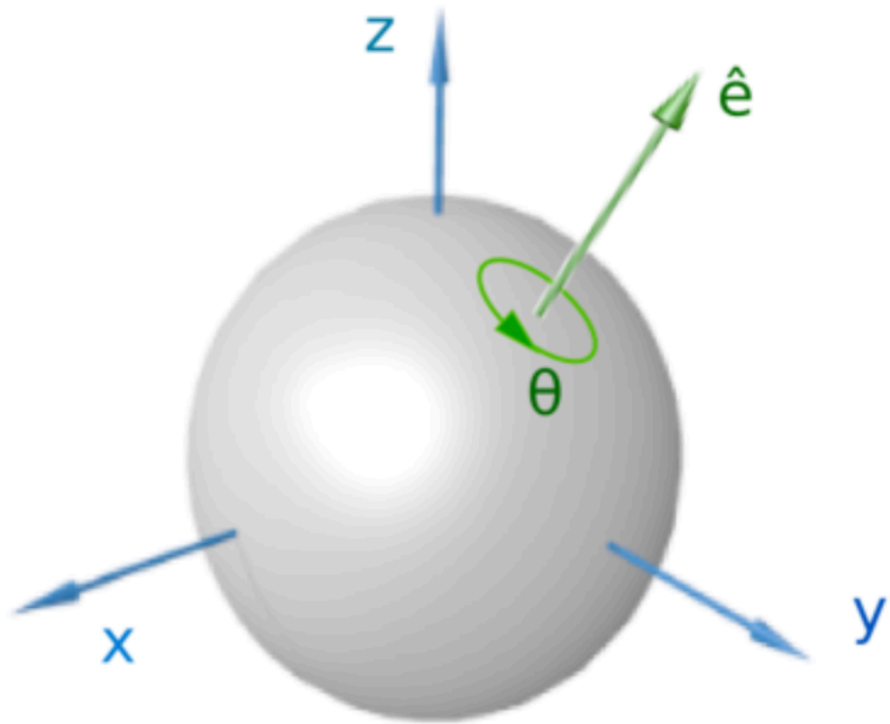
Hence, $x_{\star}^{\mathrm{v}}$ can be expressed in terms of x^{v} as

$$\begin{aligned} x_{\star}^{\mathrm{v}} &= (x^{\mathrm{v}} \cdot n^{\mathrm{v}})n^{\mathrm{v}} + (x^{\mathrm{v}} - (x^{\mathrm{v}} \cdot n^{\mathrm{v}})n^{\mathrm{v}}) \cos\alpha + (x^{\mathrm{v}} \times n^{\mathrm{v}}) \sin\alpha \\ &= x^{\mathrm{v}} \cos\alpha + n^{\mathrm{v}}(x^{\mathrm{v}} \cdot n^{\mathrm{v}})(1 - \cos\alpha) - (n^{\mathrm{v}} \times x^{\mathrm{v}}) \sin\alpha. \end{aligned} \tag{3.12}$$

Denoting the rotated coordinate frame the u-frame and using the equivalence between $x_{\star}^{\mathrm{v}}$ and x^{u} as shown in Example 3.2, this implies that

$$x^{\mathrm{u}} = x^{\mathrm{v}} \cos\alpha + n^{\mathrm{v}}(x^{\mathrm{v}} \cdot n^{\mathrm{v}})(1 - \cos\alpha) - (n^{\mathrm{v}} \times x^{\mathrm{v}}) \sin\alpha. \tag{3.13}$$

This equation is commonly referred to as the *rotation formula* or *Euler's formula* [135]. Note that the combination of n and α, or $\eta = n\alpha$, is denoted as the *rotation vector* or the *axis-angle parameterization*.



- Visualizing a rotation represented by an Euler axis and angle.

Extension of Euler's formula

A Euclidean vector such as $(2, 3, 4)$ or (a_x, a_y, a_z) can be rewritten as $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ or $a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors representing the three Cartesian axes. A rotation through an angle of θ around the axis defined by a unit vector

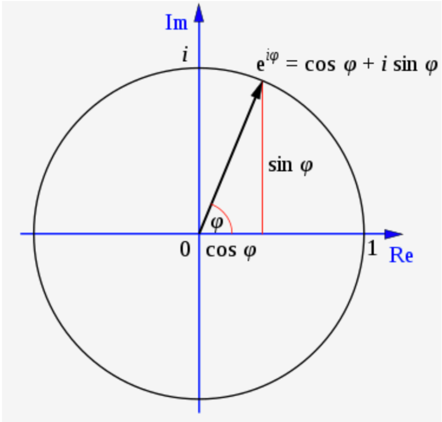
$$\vec{u} = (u_x, u_y, u_z) = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

can be represented by a quaternion. This can be done using an extension of Euler's formula:

$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}$$

Recall: Euler's formula:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$



Inverse and Composition

$$\mathbf{q}^{-1} = e^{-\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} - (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}.$$

It follows that conjugation by the product of two quaternions is the composition of conjugations by these quaternions: If $\mathbf{p}$ and $\mathbf{q}$ are unit quaternions, then rotation (conjugation) by $\mathbf{pq}$ is

$$\mathbf{pq}\vec{v}(\mathbf{pq})^{-1} = \mathbf{pq}\vec{v}\mathbf{q}^{-1}\mathbf{p}^{-1} = \mathbf{p}(\mathbf{q}\vec{v}\mathbf{q}^{-1})\mathbf{p}^{-1},$$

which is the same as rotating (conjugating) by $\mathbf{q}$ and then by $\mathbf{p}$. The scalar component of the result is necessarily zero.

Euler Angle

Rotation can also be defined as a consecutive rotation around three axes in terms of so-called *Euler angles*. We use the convention (z, y, x) which first rotates an angle ψ around the z-axis, subsequently an angle θ around the y-axis and finally an angle ϕ around the x-axis. These angles are illustrated in Figure 3.5. Assuming that the v-frame is rotated by (ψ, θ, ϕ) with respect to the u-frame as illustrated in this figure, the rotation matrix R^{uv} is given by

$$\begin{aligned} R^{\mathrm{uv}} &= R^{\mathrm{uv}}(e_1,\phi)R^{\mathrm{uv}}(e_2,\theta)R^{\mathrm{uv}}(e_3,\psi) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{pmatrix}, \end{aligned} \tag{3.20}$$

where we make use of the notation introduced in (3.17) and the following definition of the unit vectors

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^{\mathsf{T}}, \qquad e_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^{\mathsf{T}}, \qquad e_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^{\mathsf{T}}. \tag{3.21}$$

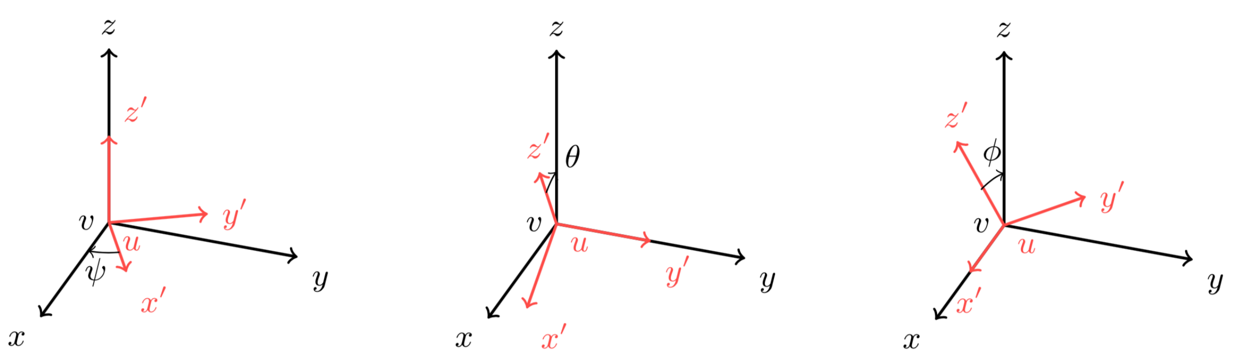


Figure 3.5: Definition of Euler angles as used in this work with left: rotation ψ around the z-axis, middle: rotation θ around the y-axis and right: rotation ϕ around the x-axis.

Pitch, roll, and yaw

The ψ, θ, ϕ angles are also often referred to as yaw (or heading), pitch and roll, respectively. Furthermore, roll and pitch together are often referred to as inclination.

Gimbal Lock

Similar to the rotation vector, Euler angles parametrize orientation as a three-dimensional vector. Euler angle representations are not unique descriptions of a rotation for two reasons. First, due to wrapping of the Euler angles, the rotation $(0, 0, 0)$ is for instance equal to $(0, 0, 2\pi k)$ for any integer k. Furthermore, setting $\theta = \frac{\pi}{2}$ in (3.20), leads to

$$
\begin{aligned}
R^{\text{uv}} &= \begin{pmatrix} 0 & 0 & -1 \\ \sin\phi\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ \cos\phi\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\psi - \sin\phi\cos\psi & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & -1 \\ \sin(\phi - \psi) & \cos(\phi - \psi) & 0 \\ \cos(\phi - \psi) & -\sin(\phi - \psi) & 0 \end{pmatrix}.
\end{aligned} \tag{3.22}
$$

Hence, only the rotation $\phi - \psi$ can be observed. Because of this, for example the rotations $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2}, -\frac{\pi}{2})$, $(\pi, \frac{\pi}{2}, \frac{\pi}{2})$ are all three equivalent. This is called *gimbal lock* [31].

- We will analyze further using quaternions next.

Exponential an skew symmetric matrix, we get a rotation matrix

Recall:

Given a square matrix $X \in \mathbb{R}^{n \times n}$,
the exponential of X is given by the absolute convergent power series

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

What is a rigid motion/transformation?

A rigid transformation is formally defined as a transformation that, when acting on any vector **v**, produces a transformed vector $T(\mathbf{v})$ of the form

$$T(\mathbf{v}) = R\,\mathbf{v} + \mathbf{t}$$

where $R^\mathsf{T} = R^{-1}$ (i.e., R is an orthogonal transformation), and **t** is a vector giving the translation of the origin.

A proper rigid transformation has, in addition,

$$\det(R) = 1$$

which means that R does not produce a reflection, and hence it represents a rotation (an orientation-preserving orthogonal transformation). Indeed, when an orthogonal transformation matrix produces a reflection, its determinant is -1.

Note: Rotation and Translation do not commute

$$T(\mathbf{v}) = R\,\mathbf{v} + \mathbf{t}$$

- This can be viewed as first rotating the vector v by using rotation matrix R, then translate by t.
- What if we first translate the vector v by t and then do the rotation by R?

Key property of any rigid motion: preserves the distance!

$$d(g(\mathbf{X}), g(\mathbf{Y}))^2 = d(\mathbf{X}, \mathbf{Y})^2.$$

$$\begin{aligned} d(\mathbf{X}, \mathbf{Y})^2 &= (X_1 - Y_1)^2 + (X_2 - Y_2)^2 + \ldots + (X_n - Y_n)^2 \\ &= (\mathbf{X} - \mathbf{Y}) \cdot (\mathbf{X} - \mathbf{Y}). \end{aligned}$$

For the Euclidean distance d and

a rigid transformation g:$\mathrm{R}^n \rightarrow \mathrm{R}^n$

Claim: The tangent plane of the Lie group SO(3) at the identity is the set of skew-symmetric matrices.

Work out details with the students on the board.

Even we are interested in n=3. All the derivations can be extended to dim=n.

$$SO(n) = \{Q \in \mathbb{R}^{n\times n} : \ Q^\top = Q^{-1}, \ \det(Q) = 1\}.$$

The algebra of skew-symmetric matrices is denoted by

$$so(n) = \{S \in \mathbb{R}^{n\times n} : \ S^\top = -S\}$$

so(n) is called the Lie algebra of the Lie group SO(n).

The set of rigid motions SE(n) also form a Lie Group

$$SE(n) = \left\{ \begin{bmatrix} Q & \mathbf{u} \\ \mathbf{0} & 1 \end{bmatrix} : Q \in SO(n), \mathbf{u} \in \mathbb{R}^{n\times 1} \right\}.$$

While $SE(n)$ describes configurations, its Lie algebra $se(n)$, defined by

$$se(n) = \left\{ \begin{bmatrix} S & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} : S \in so(n), \mathbf{v} \in \mathbb{R}^{n\times 1} \right\},$$

- Note: Here we used the homogeneous representation of rotation and translation.

Geometric aspect of the exponential and logarithm

To any vector $\mathbf{u} = (u_1,\ u_2,\ u_3) \in \mathbb{R}^3$ one can associate the 3×3 skew-symmetric matrix

$$S_{\mathbf{u}} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

It is easy to see that, for every $\mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \times \mathbf{v} = S_{\mathbf{u}}\,\mathbf{v},$$

where $\times$ stands for the cross product.

Now consider the following problem: given a unit vector $\mathbf{u} \in \mathbb{R}^3$ and an angle $\theta \in \mathbb{R}$, find the rotation matrix R that rotates any vector through the angle θ about an axis with direction $\mathbf{u}$.

The matrix exponential gives the elegant solution:

$$R = e^{S_{\mathbf{u}}\theta}.$$

- Moreover, we have a closed formula for it:

$$e^{S_{\mathbf{u}}\theta} = I + \sin\theta\, S_{\mathbf{u}} + (1 - \cos\theta)\, S_{\mathbf{u}}^2.$$

- The above is the famous **Rodrigues Formula.**

Derivation of Rodrigues Formula

- Work out the details with the students.

- ***The power of Exp and Log of matrices!***

Conversely, given $R \in SO(3)$ (with no negative eigenvalues) consider the problem of finding the axis direction $\mathbf{u}$ and the angle θ of rotation. Using the matrix exponential,
we can formulate this problem as follows:
determine a unitary vector $\mathbf{u}$ and $\theta \in]-\pi, \pi[$ such that

$$R = e^{S_{\mathbf{u}}\theta}.$$

The matrix logarithm provides the simple answer

$$S_{\mathbf{u}}\theta = \log(R),$$

or equivalently,

$$S_{\mathbf{u}} = \frac{1}{\theta}\log(R), \quad \text{whenever } \theta \neq 0.$$

A widely used formula for computing the logarithm of a 3×3 rotation matrix is

$$\log R = \frac{\theta}{2 \sin \theta}(R^\top - R),$$

where θ satisfies $1 + 2\cos\theta = \text{trace}(R), \theta \neq 0, -\pi < \theta < \pi$.

When $\theta = 0$ one has the trivial case $R = I$ and $\log R = 0$.

- Note:

How to calculate the angle?

- Taking the norm both sides of

$$S_{\mathbf{u}} = \frac{1}{\theta} \log(R),$$

- We get

$$\| S_{\mathbf{u}} \| \, |\theta| = \| \log(R) \|,$$

and, since $\| S_{\mathbf{u}} \| = 1$, the angle of rotation is related with the norm of $\log(R)$ by

$$|\theta| = \| \log(R) \|.$$

Summarize

- This means that the direction of the rotation axis is given by the logarithm of the matrix associated with u.

- This relationship between skew-symmetric and rotation matrices by means of exponentials and logarithms are the key to explain the importance of these matrix functions in rigid motions and robotics. There are many other geometric problems involving exponentials and logarithms of matrices. For example, my student Tum and I was working on the design of trajectories for UAVs in an indoor environments (meaning assuming without GPS).

Claim: Given a unit quaternion, we can define a rotation.

What is the matrix for R_q?

- Ans: It is the matrix representation of $\mathbf{R_{q.}}$
- How to find it?
- Ans: Let $\mathbf{R_q}$ acts on the basis {1, i, j, k}

$$\mathbf{R} = \begin{bmatrix} 1 - 2s(q_j^2 + q_k^2) & 2s(q_i q_j - q_k q_r) & 2s(q_i q_k + q_j q_r) \\ 2s(q_i q_j + q_k q_r) & 1 - 2s(q_i^2 + q_k^2) & 2s(q_j q_k - q_i q_r) \\ 2s(q_i q_k - q_j q_r) & 2s(q_j q_k + q_i q_r) & 1 - 2s(q_i^2 + q_j^2) \end{bmatrix}$$

Here $s = ||q||^{-2}$ and if q is a unit quaternion, $s = 1$.

Recovering the axis-angle representation

The axis a and angle θ corresponding to a quaternion $\mathbf{q} = q_r + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$ can be extracted via:

$$(a_x, a_y, a_z) = \frac{(q_i, q_j, q_k)}{\sqrt{q_i^2 + q_j^2 + q_k^2}}$$

$$\theta = 2\,\mathrm{atan2}\left(\sqrt{q_i^2 + q_j^2 + q_k^2}, q_r\right),$$

where $\mathrm{atan2}$ is the two-argument arctangent. Care should be taken when the quaternion approaches a real quaternion, since due to degeneracy the axis of an identity rotation is not well-defined.

Differentiation with respect to the rotation quaternion

The rotated quaternion $\mathbf{p'}=\mathbf{q}\,\mathbf{p}\,\mathbf{q}^*$ needs to be differentiated with respect to the rotating quaternion $\mathbf{q}$, when the rotation is estimated from numerical optimization. The estimation of rotation angle is an essential procedure in 3D object registration or camera calibration. The derivative can be represented using the Matrix Calculus notation.

$$\frac{\partial \mathbf{p}'}{\partial \mathbf{q}} \equiv \left[\frac{\partial \mathbf{p}'}{\partial q_0}, \frac{\partial \mathbf{p}'}{\partial q_x}, \frac{\partial \mathbf{p}'}{\partial q_y}, \frac{\partial \mathbf{p}'}{\partial q_z}\right]$$

$$= \left[\mathbf{pq} - (\mathbf{pq})^*, (\mathbf{pqi})^* - \mathbf{pqi}, (\mathbf{pqj})^* - \mathbf{pqj}, (\mathbf{pqk})^* - \mathbf{pqk}\right].$$

Types of Matrix Derivatives

Types	Scalar	Vector	Matrix
Scalar	$\frac{\partial y}{\partial x}$	$\frac{\partial \mathbf{y}}{\partial x}$	$\frac{\partial \mathbf{Y}}{\partial x}$
Vector	$\frac{\partial y}{\partial \mathbf{x}}$	$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$	
Matrix	$\frac{\partial y}{\partial \mathbf{X}}$		

Will be covered in Math 173, Advanced linear Algebra.

We also can add a scalar to a vector and find inverse of a vector!

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a + \vec{v}.$$

$$a + \vec{v} = (a, \vec{0}) + (0, \vec{v}).$$

$$(s + \vec{v})^{-1} = \frac{(s + \vec{v})^*}{\|s + \vec{v}\|^2} = \frac{s - \vec{v}}{s^2 + \|\vec{v}\|^2}$$

Now we can multiply two vectors in R^3 and in R^4 !
First define it in R^3
by viewing them as pure imaginary quaternions

We can express quaternion multiplication in the modern language of vector cross and dot products (which were actually inspired by the quaternions in the first place [6]). When multiplying the vector/imaginary parts, in place of the rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ we have the quaternion multiplication rule:

$$\vec{v}\vec{w} = \vec{v} \times \vec{w} - \vec{v} \cdot \vec{w},$$

where:

- $\vec{v}\vec{w}$ is the resulting quaternion,
- $\vec{v} \times \vec{w}$ is vector cross product (a vector),
- $\vec{v} \cdot \vec{w}$ is vector scalar product (a scalar).

Quaternion multiplication is noncommutative (because of the cross product, which anti-commutes), while scalar–scalar and scalar–vector multiplications commute. From these rules it follows immediately that (see details):

$$(s + \vec{v})(t + \vec{w}) = (st - \vec{v} \cdot \vec{w}) + (s\vec{w} + t\vec{v} + \vec{v} \times \vec{w}).$$

Quaternions are extension of complex numbers

The complex numbers can be defined by introducing an abstract symbol $\mathbf{i}$ which satisfies the usual rules of algebra and additionally the rule $\mathbf{i}^2 = -1$. This is sufficient to reproduce all of the rules of complex number arithmetic: for example:

$$(a + b\mathbf{i})(c + d\mathbf{i}) = ac + ad\mathbf{i} + b\mathbf{i}c + b\mathbf{i}d\mathbf{i}$$
$$= ac + ad\mathbf{i} + bc\mathbf{i} + bd\mathbf{i}^2 = (ac - bd) + (bc + ad)\mathbf{i}.$$

Multiply two quaternions:

$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(e + f\mathbf{i} + g\mathbf{j} + h\mathbf{k}) =$$
$$(ae - bf - cg - dh) + (af + be + ch - dg)\mathbf{i} + (ag - bh + ce + df)\mathbf{j} + (ah + bg - cf + de)\mathbf{k}.$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1$$

Notations for using multiple frames

- Pay attention to the subscripts!

$$q = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} q_0 \\ q_v \end{pmatrix}, \qquad q \in \mathbb{R}^4, \qquad \|q\|_2 = 1.$$

A rotation can be defined using unit quaternions as

$$\bar{x}^{\mathrm{u}} = q^{\mathrm{uv}} \odot \bar{x}^{\mathrm{v}} \odot (q^{\mathrm{uv}})^{\mathsf{c}},$$

where $\cdot^{\mathsf{c}}$ denotes the quaternion conjugate, defined as

$$q^{\mathsf{c}} = \begin{pmatrix} q_0 & -q_v^{\mathsf{T}} \end{pmatrix}^{\mathsf{T}},$$

and $\bar{x}^{\mathrm{v}}$ denotes the quaternion representation of x^{v} as

$$\bar{x}^{\mathrm{v}} = \begin{pmatrix} 0 & (x^{\mathrm{v}})^{\mathsf{T}} \end{pmatrix}^{\mathsf{T}}.$$

Caution: $-q$ describes the same orientation.

- Be able to change quaternion multiplication to matrix/vector multiplication fluently.

$$p \odot q = \begin{pmatrix} p_0 q_0 - p_v \cdot q_v \\ p_0 q_v + q_0 p_v + p_v \times q_v \end{pmatrix} = p^{\mathsf{L}} q = q^{\mathsf{R}} p,$$

where

$$p^{\mathsf{L}} \triangleq \begin{pmatrix} p_0 & -p_v^{\mathsf{T}} \\ p_v & p_0 \mathcal{I}_3 + [p_v \times] \end{pmatrix}, \qquad q^{\mathsf{R}} \triangleq \begin{pmatrix} q_0 & -q_v^{\mathsf{T}} \\ q_v & q_0 \mathcal{I}_3 - [q_v \times] \end{pmatrix}.$$

$$
\begin{aligned}
\bar{x}^{\mathrm{u}} &= (q^{\mathrm{uv}})^{\mathsf{L}} (q^{\mathrm{vu}})^{\mathsf{R}} \bar{x}^{\mathrm{v}} \\
&= \begin{pmatrix} q_0 & -q_v^{\mathsf{T}} \\ q_v & q_0 \mathcal{I}_3 + [q_v \times] \end{pmatrix} \begin{pmatrix} q_0 & q_v^{\mathsf{T}} \\ -q_v & q_0 \mathcal{I}_3 + [q_v \times] \end{pmatrix} \begin{pmatrix} 0 \\ x^{\mathrm{v}} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0_{1\times 3} \\ 0_{3\times 1} & q_v q_v^{\mathsf{T}} + q_0^2 \mathcal{I}_3 + 2 q_0 [q_v \times] + [q_v \times]^2 \end{pmatrix} \begin{pmatrix} 0 \\ x^{\mathrm{v}} \end{pmatrix}.
\end{aligned}
$$

Compare with

$$
R^{\mathrm{uv}}(n^{\mathrm{v}}, \alpha) = \mathcal{I}_3 - \sin\alpha [n^{\mathrm{v}} \times] + (1 - \cos\alpha)[n^{\mathrm{v}} \times]^2. \tag{3.17}
$$

it can be recognized that if we choose

$$
q^{\mathrm{uv}}(n^{\mathrm{v}}, \alpha) = \begin{pmatrix} \cos\frac{\alpha}{2} \\ -n^{\mathrm{v}} \sin\frac{\alpha}{2} \end{pmatrix},
$$

the two rotation formulations are equivalent since

$$\bar{x}^{\mathrm{u}} = \begin{pmatrix} 1 & 0_{1\times 3} \\ 0_{3\times 1} & \mathcal{I}_3 - 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}[n^{\mathrm{v}}\times] + 2\sin^2\frac{\alpha}{2}[n^{\mathrm{v}}\times]^2 \end{pmatrix} \begin{pmatrix} 0 \\ x^{\mathrm{v}} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0_{1\times 3} \\ 0_{3\times 1} & \mathcal{I}_3 - \sin\alpha[n^{\mathrm{v}}\times] + (1-\cos\alpha)\,[n^{\mathrm{v}}\times]^2 \end{pmatrix} \begin{pmatrix} 0 \\ x^{\mathrm{v}} \end{pmatrix}. \tag{3.31}$$

Here, we made use of standard trigonometric relations and the fact that since $\|n^{\mathrm{v}}\|_2 = 1$, $n^{\mathrm{v}}\,(n^{\mathrm{v}})^{\mathsf{T}} = \mathcal{I}_3 + [n^{\mathrm{v}}\times]^2$. Hence, it can be concluded that q^{uv} can be expressed in terms of α and n^{v} as in (3.30).

Equivalently, $q^{\text{uv}}(n^{\text{v}}, \alpha)$ can also be written as

$$q^{\text{uv}}(n^{\text{v}}, \alpha) = \exp(-\tfrac{\alpha}{2}\bar{n}^{\text{v}}) = \sum_{k=0}^{\infty} \tfrac{1}{k!} \left(-\tfrac{\alpha}{2}\bar{n}^{\text{v}}\right)^k, \tag{3.32}$$

where

$$(\bar{n}^{\text{v}})^0 = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^{\mathsf{T}}, \tag{3.33a}$$

$$(\bar{n}^{\text{v}})^1 = \begin{pmatrix}0 & (n^{\text{v}})^{\mathsf{T}}\end{pmatrix}^{\mathsf{T}}, \tag{3.33b}$$

$$(\bar{n}^{\text{v}})^2 = \bar{n}^{\text{v}} \odot \bar{n}^{\text{v}} = \begin{pmatrix}-\|n^{\text{v}}\|_2^2 & 0_{3\times1}\end{pmatrix}^{\mathsf{T}} = \begin{pmatrix}-1 & 0_{3\times1}\end{pmatrix}^{\mathsf{T}}, \tag{3.33c}$$

$$(\bar{n}^{\text{v}})^3 = \begin{pmatrix}0 & -(n^{\text{v}})^{\mathsf{T}}\end{pmatrix}^{\mathsf{T}}, \tag{3.33d}$$

This leads to

$$\begin{aligned} q^{\text{uv}}(n^{\text{v}}, \alpha) &= \exp(-\tfrac{\alpha}{2}\bar{n}^{\text{v}}) = \sum_{k=0}^{\infty} \tfrac{1}{k!} \left(-\tfrac{\alpha}{2}\bar{n}^{\text{v}}\right)^k \\ &= \begin{pmatrix} 1 - \tfrac{1}{2!}\tfrac{\alpha^2}{4} + \tfrac{1}{4!}\tfrac{\alpha^4}{16} - \dots \\ -\tfrac{\alpha}{2}n^{\text{v}} + \tfrac{1}{3!}\tfrac{\alpha^3}{8}n^{\text{v}} - \tfrac{1}{5!}\tfrac{\alpha^5}{32}n^{\text{v}} + \dots \end{pmatrix} \\ &= \begin{pmatrix} \cos\tfrac{\alpha}{2} \\ -n^{\text{v}}\sin\tfrac{\alpha}{2} \end{pmatrix}. \end{aligned} \tag{3.34}$$

Note the similarity to (3.18) and (3.19). The reason why both rotation matrices and unit quaternions can be described in terms of an exponential of a rotation vector will be discussed in §3.3.1.

Cautions

- Some exiting estimation algorithms typically assume that the unknown states and parameters are represented in Euclidean space.
  - For instance, they assume that the subtraction of two orientations gives information about the difference in orientation and that the addition of two orientations is again a valid orientation.
- If you work in a parameterizing space, you still need to be careful.
  - For instance, due to wrapping and gimbal lock, subtraction of Euler angles and rotation vectors can result in large numbers even in cases when the rotations are similar.
- Subtraction of unit quaternions and rotation matrices do not in general result in a valid rotation.
- The equality constraints on the norm of unit quaternions and on the determinant and the orthogonally of rotation matrices are typically hard to include in the estimation algorithms.
- We will discuss some methods later to represent orientation in estimation algorithms that deals with the issues described above.
  - frequently used correct representations and algorithms

- we will also discuss some alternative methods to parametrize orientation for estimation purposes.

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{C(t+\delta t) + C(t)}{\delta t}$$

Since $C(t+\delta t)$ also represents a rotation, we choose to write is as

$$C(t+\delta t)C(t)A(t)$$

for some rotation matrix $A(t)$.

Recall that rotations about the x, y, and z axes can be written respectively as

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \qquad R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad R_z = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_x R_y R_z$$

for some ϕ, θ, and φ, which are then known as Tait–Bryan angles. Multiplying R out yields

$$\begin{bmatrix} \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta\cos\varphi - \sin\varphi\cos\phi & \sin\phi\sin\varphi\sin\theta + \cos\phi\cos\varphi & \sin\phi\cos\theta \\ \sin\phi\sin\varphi + \sin\theta\cos\phi\cos\varphi & -\sin\phi\cos\varphi + \sin\varphi\sin\theta\cos\phi & \cos\phi\cos\theta \end{bmatrix}.$$

If ϕ, θ, and φ approach zero, we can make a small angle approximation yielding

$$R = R_x R_y R_z = \begin{bmatrix} 1 & \varphi & -\theta \\ -\varphi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix}.$$

Since δt is small, we can write

$$A(t) = I + \delta\Psi(t)$$

where

$$\delta\Psi(t) = \begin{bmatrix} 0 & -\delta\varphi & \delta\theta \\ \delta\varphi & 0 & -\delta\phi \\ -\delta\theta & \delta\phi & 1 \end{bmatrix}.$$

Thus, substituting into our original forward-difference yields

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}t} &= \lim_{\delta t \to 0} \frac{C(t)(I + \delta\Psi(t)) - C(t)}{\delta t} \\ &= C(t) \lim_{\delta t \to 0} \frac{\mathrm{d}\Psi}{\mathrm{d}t} \\ &= C(t)\Omega(t) \end{aligned}$$

where

$$\Omega(t) = \begin{bmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{bmatrix}.$$

This is the skew-symmetric form of the angular velocity vector $\boldsymbol{\omega}(t)$, which we can acquire periodically from the gyroscope. Thus, we are interested in solving the differential equation

$$\dot{C}(t) = C(t)\Omega(t).$$

This has the solution

$$C(t) = C(0) \cdot \exp\left(\int_0^t \Omega(t)\ \mathrm{d}t\right).$$

Correct ways we use

- Recall the set of rotations is a Lie group, so there exists an exponential map from a corresponding Lie algebra.
- We use exponential map from so(3) to SO(3) and logarithm from SO(3) to so(3).
- We represent orientations on SO(3) using unit quaternions or rotation matrices,
- We represent orientation deviations using rotation vectors on R3 (Key! This mimic how we deal with rotation in R2 using Euler angle.)

Specifically,

we encode an orientation q_t^{nb} in terms of a *linearization point* parametrized either as a unit quaternion $\tilde{q}_t^{\text{nb}}$ or as a rotation matrix $\tilde{R}_t^{\text{nb}}$ and an *orientation deviation* using a rotation vector η_t. Assuming that the orientation deviation is expressed in the body frame b,[1]

$$q_t^{\text{nb}} = \tilde{q}_t^{\text{nb}} \odot \exp\left(\frac{\bar{\eta}_t^{\text{b}}}{2}\right), \qquad R_t^{\text{nb}} = \tilde{R}_t^{\text{nb}} \exp\left([\eta_t^{\text{b}} \times]\right), \tag{3.35}$$

where analogously to (3.34) and (3.19),

$$\exp(\bar{\eta}) = \begin{pmatrix} \cos \|\eta\|_2 \\ \frac{\eta}{\|\eta\|_2} \sin \|\eta\|_2 \end{pmatrix}, \tag{3.36a}$$

$$\exp([\eta\times]) = \mathcal{I}_3 + \sin\left(\|\eta\|_2\right) \left[\frac{\eta}{\|\eta\|_2}\times\right] + \left(1 - \cos\left(\|\eta\|_2\right)\right) \left[\frac{\eta}{\|\eta\|_2}\times\right]^2. \tag{3.36b}$$

Gyroscope measurement models (Later)

As discussed in §2.2, the gyroscope measures the angular velocity $\omega_{\mathrm{ib}}^{\mathrm{b}}$ at each time instance t. However, as shown in §2.4, its measurements are corrupted by a slowly time-varying bias $\delta_{\omega,t}$ and noise $e_{\omega,t}$. Hence, the gyroscope measurement model is given by

$$y_{\omega,t} = \omega_{\mathrm{ib},t}^{\mathrm{b}} + \delta_{\omega,t}^{\mathrm{b}} + e_{\omega,t}^{\mathrm{b}}. \tag{3.41}$$