Lecture 6

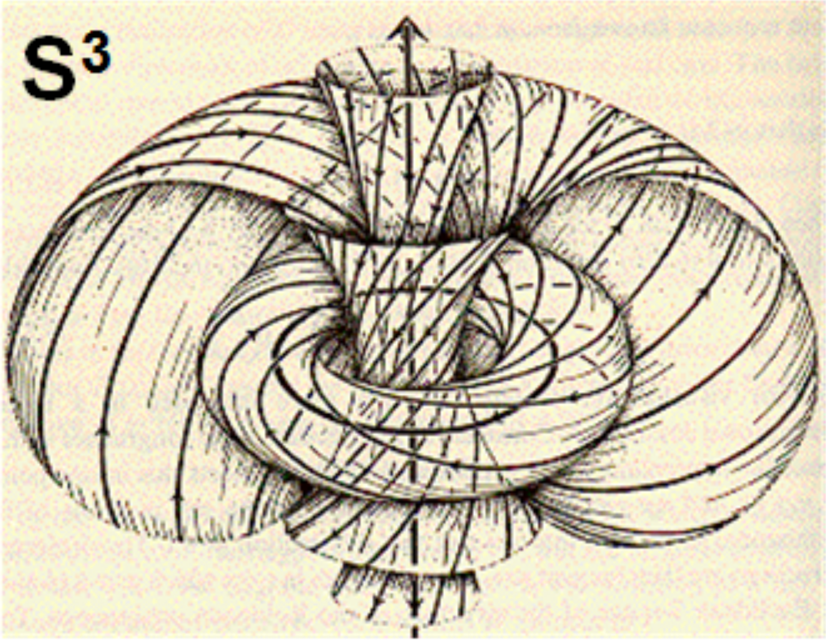
Math 178

Prof. Weiqing Gu

S^3 is a double cover of SO(3)

- Last time: Very pair of unit quaternions q & -q represents a rotation.
- The set of unit quaternions form a sphere S^3 in R^4.
- We can show that S^3 is a double cover of SO(3).

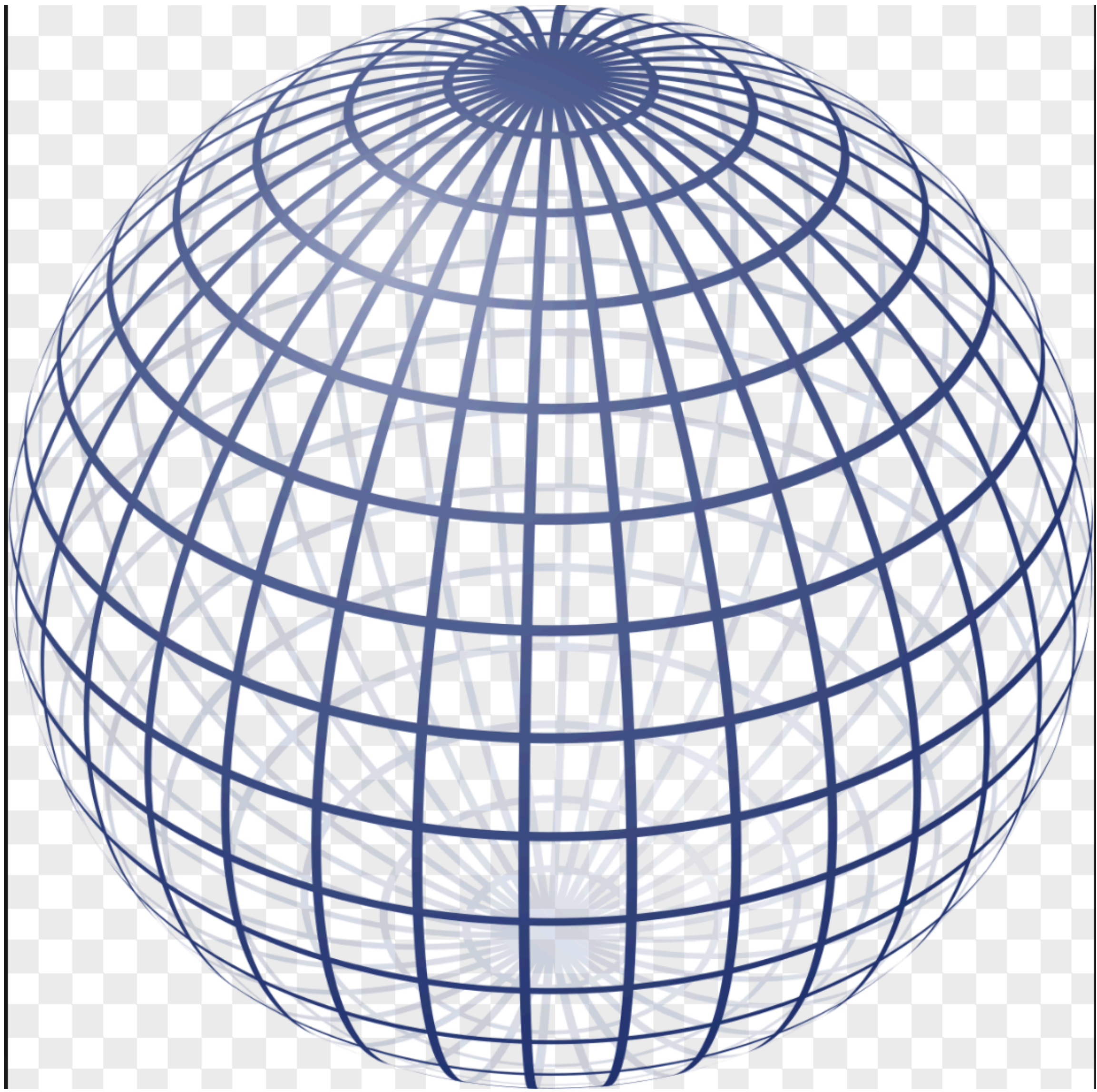
How does S^3 look like?



By identify antipodal points

$$M = SO(3) = RP^3$$

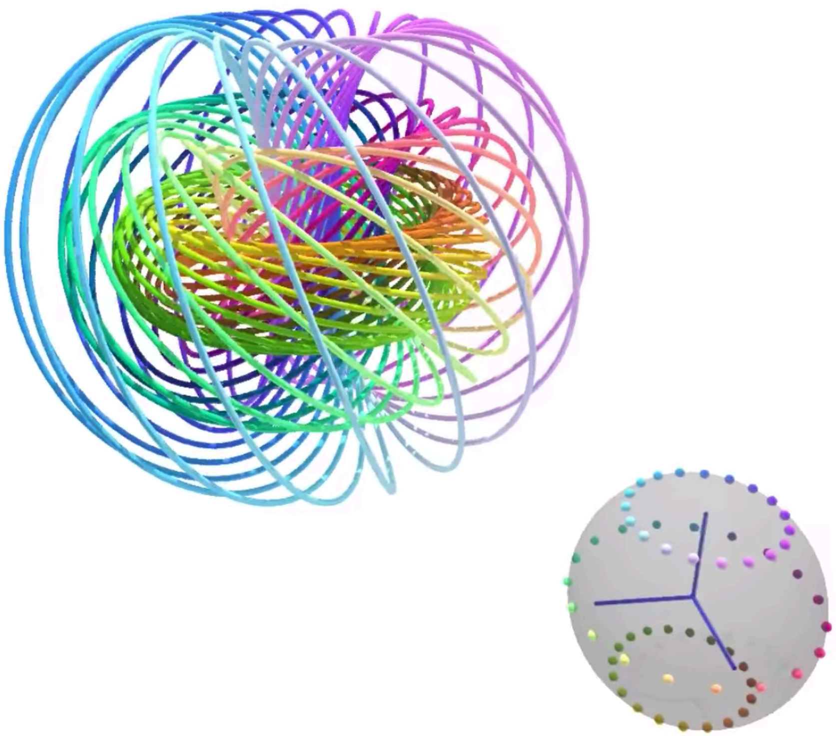
Recall: There are a lot of circles on S^2



There are also a lots of circles on S^3

Stereographic projection of the hypersphere's parallels (red), meridians (blue) and hypermeridians (green). Because this projection is conformal, the curves intersect each other orthogonally (in the yellow points) as in 4D. All curves are circles: the curves that intersect ⟨0,0,0,1⟩ have infinite radius (= straight line). In this picture, the whole 3D space maps the *surface* of the hypersphere, whereas in the previous picture[*clarification needed*] the 3D space contained the *shadow* of the bulk hypersphere.

Hopf Fibration



- https://www.youtube.com/watch?v=AKotMPGFJYk

Rigid Body Kinematics

- https://www.seas.upenn.edu/~meam620/slides/kinematicsI.pdf

The important subgroups of SE(3)

Subgroup	Notation	Definition	Significance
The group of rotations in three dimensions	$SO(3)$	The set of all proper orthogonal matrices. $SO(3)=\{\mathbf{R} \mid \mathbf{R} \in R^{3\times 3}, \mathbf{R}^T\mathbf{R}=\mathbf{R}\mathbf{R}^T=\mathbf{I}\}$	All spherical displacements. Or the set of all displacements that can be generated by a spherical joint (*S*-pair).
Special Euclidean group in two dimensions	$SE(2)$	The set of all 3×3 matrices with the structure: $\begin{bmatrix} \cos\theta & \sin\theta & r_x \\ -\sin\theta & \cos\theta & r_y \\ 0 & 0 & 1 \end{bmatrix}$ where θ, r_x, and r_y are real numbers.	All planar displacements. Or the set of displacements that can be generated by a planar pair (*E*-pair).
The group of rotations in two dimensions	$SO(2)$	The set of all 2×2 proper orthogonal matrices. They have the structure: $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where θ is a real number.	All rotations in the plane, or the set of all displacements that can be generated by a single revolute joint (*R*-pair).

The important subgroups of SE(3) (continue)

The group of translations in n dimensions.	$T(n)$	The set of all $n \times 1$ real vectors with vector addition as the binary operation.	All translations in n dimensions. $n = 2$ indicates planar, $n = 3$ indicates spatial displacements.
The group of translations in one dimension.	$T(1)$	The set of all real numbers with addition as the binary operation.	All translations parallel to one axis, or the set of all displacements that can be generated by a single prismatic joint (*P*-pair).
The group of cylindrical displacements	$SO(2) \times T(1)$	The Cartesian product of $SO(2)$ and $T(1)$	All rotations in the plane and translations along an axis perpendicular to the plane, or the set of all displacements that can be generated by a cylindrical joint (*C*-pair).
The group of screw displacements	$H(1)$	A one-parameter subgroup of $SE(3)$	All displacements that can be generated by a helical joint (*H*-pair).

The Group of Rotations

A rigid body B is said to rotate relative to another rigid body A, when a point of B is always fixed in $\{A\}$. Attach the frame $\{B\}$ so that its origin O' is at the fixed point. The vector ${}^{A}\mathbf{r}^{O'}$ is equal to zero in the homogeneous transformation in Equation (1).

$$
{}^{A}\mathbf{A}_{B} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & \mathbf{0}_{3\times 1} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
$$

The set of all such displacements, also called *spherical displacements*, can be easily seen to form a subgroup of $SE(3)$.

If we compose two rotations, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, the product is given by:

$$
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & \mathbf{0}_{3\times 1} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \times \left[\begin{array}{c|c} {}^{B}\mathbf{R}_{C} & \mathbf{0}_{3\times 1} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
$$

$$
= \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} & \mathbf{0}_{3\times 1} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
$$

Notice that only the 3×3 submatrix of the homogeneous transformation matrix plays a role in describing rotations. Further, the binary operation of multiplying 4×4 homogoneous transformation matrices reduces to the binary operation of multiplying the corresponding 3×3 submatrices. Thus, we can simply use 3×3 rotation matrices to represent spherical displacements. This subgroup, is called the special orthogonal group in three dimensions, or simply $SO(3)$:

$$
SO(3) = \left\{ \mathbf{R} \;\middle|\; \mathbf{R} \in R^{3\times 3},\ \mathbf{R}^{T}\mathbf{R} = \mathbf{R}\mathbf{R}^{T} = \mathbf{I} \right\} \tag{4}
$$

Locally exponential down to form data manifold

- Suppose we have data points in R^n and clustered in different areas. In each area find a center c of the data points and view them as a vectors in R^n. Use exponential map to mape this tangent space at c to form a manifold.
- Cover the data using those areas and then then exponential down all of them.
- The collections of those exponential down curved pieces could form a manifold?

Decompose a Rotation to 3 Successive Rotations

It is well known that any rotation can be decomposed into three finite successive rotations, each about a different axis than the preceding rotation. The three rotation angles, called Euler angles, completely describe the given rotation. The basic idea is as follows. If we consider any two reference frames $\{A\}$ and $\{B\}$, and the rotation matrix ${}^{A}\mathbf{R}_{B}$, we can construct two intermediate reference frames $\{M\}$ and $\{N\}$, so that

$$ {}^{A}\mathbf{R}_{B} = {}^{A}\mathbf{R}_{M} \times {}^{M}\mathbf{R}_{N} \times {}^{N}\mathbf{R}_{B} $$

where

1. The rotation from $\{A\}$ to $\{M\}$ is a rotation about the x axis (of $\{A\}$) through ψ;
2. The rotation from $\{M\}$ to $\{N\}$ is a rotation about the y axis (of $\{M\}$) through ϕ; and
3. The rotation from $\{N\}$ to $\{B\}$ is a rotation about the z axis (of $\{N\}$) through θ.

$$ {}^{A}\mathbf{R}_{B} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \times \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Thus any rotation can be viewed as a composition of these three elemental rotations except for rotations at which the Euler angle representation is singular[1]. This in turn means all rotations in an open neighborhood in $SO(3)$ can be described by three real numbers (coordinates). With a little work it can be shown that there is a 1-1, continuous map from $SO(3)$ onto an open set in R^3. This gives $SO(3)$ the structure of a three-dimensional differentiable manifold, and therefore a Lie group.

The rotations in the plane, or more precisely rotations about axes that are perpendicular to a plane, form a subgroup of $SO(3)$, and therefore of $SE(3)$. To see this, consider the *canonical form* of this set of rotations, the rotations about the z axis. In other words, connect the rigid bodies A and B with a revolute joint whose axis is along the z axis in Figure 1. The homogeneous transformation matrix has the form:

$$
{}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where θ is the angle of rotation. If we compose two such rotations, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, through θ_1 and θ_2 respectively, the product is given by:

$$
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) & 0 & 0 \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

All matrices in this subgroup are the same periodic function of one real variable, θ, given by:

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This subgroup is called $SO(2)$. Further, since $\mathbf{R}(\theta_1) \times \mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2)$, we can think of the subgroup as being locally isomorphic[2] to R^1 with the binary operation being addition.

The group of translations

A rigid body B is said to translate relative to another rigid body A, if we can attach reference frames to A and to B that are always parallel. The rotation matrix ${}^{A}\mathbf{R}_{B}$ equals the identity in the homogeneous transformation in Equation (1).

$${}^{A}\mathbf{A}_{B} = \left[\begin{array}{c|c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]$$

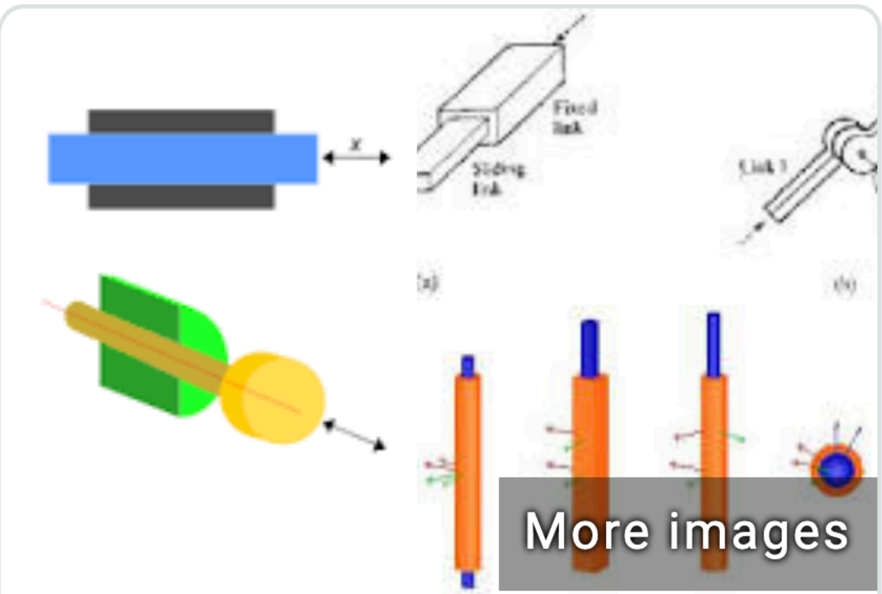
The set of all such homogeneous transformation matrices is the group of translations in three dimensions and is denoted by $T(3)$.

If we compose two translations, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, the product is given by:

$$
\begin{aligned}
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} &= \left[\begin{array}{c:c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \times \left[\begin{array}{c:c} \mathbf{I}_{3\times 3} & {}^{B}\mathbf{r}^{O''} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \\
&= \left[\begin{array}{c:c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} + {}^{B}\mathbf{r}^{O''} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
\end{aligned}
$$

Notice that only the 3×1 vector part of the homogeneous transformation matrix plays a role in describing translations. Thus we can think of a element of $T(3)$ as simply a 3×1 vector, ${}^{A}\mathbf{r}^{O'}$. Since the composition of two translations is captured by simply adding the two corresponding 3×1 vectors, ${}^{A}\mathbf{r}^{O'}$ and ${}^{B}\mathbf{r}^{O''}$, we can define the subgroup, $T(3)$, as the real vector space R^3 with the binary operation being vector addition.

Similarly, we can describe the two subgroups of $T(3)$, $T(1)$ and $T(2)$, the group of translations in one and two dimensions respectively. Because they are subgroups of $T(3)$, they are also subgroups of $T(3)$. It is worth noting that $T(1)$ consists of all translations along an axis and this is exactly the set of displacements that can be generated by connecting A and B with a single prismatic joint.



Prismatic joint

A prismatic joint provides a linear sliding movement between two bodies, and is often called a slider, as in the slider-crank linkage. A prismatic pair is also called as sliding pair. A prismatic joint can be formed with a polygonal cross-section to resist rotation. Wikipedia

The special Euclidean group in two dimensions

If we consider all rotations and translations in the plane, we get the set of all displacements that are studied in *planar kinematics*. These are also the displacements generated by the *Ebene*-pair, the planar *E*-pair. If we let the rigid body *B* translate along the *x* and *y* axis and rotate about the *z* axis relative to the frame $\{A\}$, we get the canonical set of homogeneous transformation matrices of the form:

$$
{}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & {}^{A}r_x^{O'} \\ -\sin\theta & \cos\theta & 0 & {}^{A}r_y^{O'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where θ is the angle of rotation, and ${}^{A}r_x^{O'}$ and ${}^{A}r_y^{O'}$ are the two components of translation of the origin O'. If we compose two such displacements, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, the product is given by:

$$
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & {}^{A}r_x^{O'} \\ -\sin\theta_1 & \cos\theta_1 & 0 & {}^{A}r_y^{O'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & {}^{B}r_x^{O''} \\ -\sin\theta_2 & \cos\theta_2 & 0 & {}^{B}r_y^{O''} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 & \left({}^{A}r_x^{O'} + {}^{B}r_x^{O''}\cos\theta_1 + {}^{B}r_y^{O''}\sin\theta_1\right) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & \left({}^{A}r_y^{O'} - {}^{B}r_x^{O''}\sin\theta_1 + {}^{B}r_y^{O''}\cos\theta_1\right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Because the set of matrices can be continuously parameterized by three variables, θ, ${}^{A}r_x^{O'}$, and ${}^{A}r_y^{O'}$, $SE(2)$ is a differentiable, three-dimensional manifold.

The one-parameter subgroup in SE(3)

The group of cylindrical motions is the group of motions admitted by a *cylindrical pair*, or a *C*-pair. If we let the rigid body B translate along and rotate about the z axis relative to the frame $\{A\}$, we get the canonical set of homogeneous transformation matrices of the form:

$$ {}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

where θ is the angle of rotation and k is the translation. The set of such matrices is continuously parameterized by these two variables. Thus, this subgroup is a two-dimensional Lie group. In fact, it is nothing but the Cartesian product $SO(2) \times T(1)$. Physically this means we can realize the cylindrical pair by arranging a revolute joint and a prismatic joint in series (in any order) along the same axis.

A one-dimensional subgroup is obtained by coupling the translation and the rotation so that they are proportional. The canonical homogeneous transformation matrix is of the form:

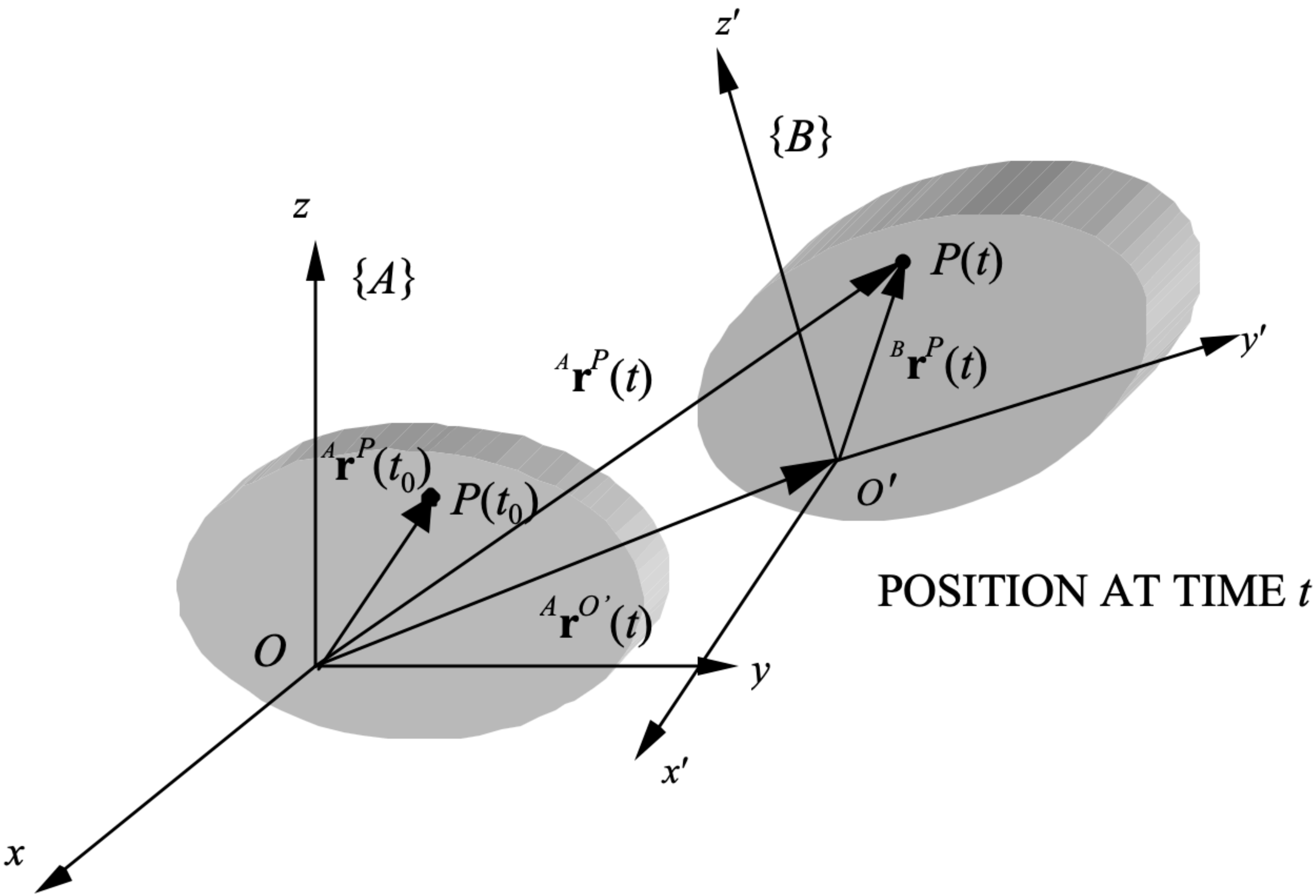
$$
{}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & h\theta \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where h is a scalar constant called the pitch. Because the displacement involves a rotation and a co-axial translation that is linearly coupled to the rotation, this displacement is called a *screw displacement*. It is exactly the displacement generated by a *helical pair*, or the *H*-pair.

$$
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & h\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & h\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 & 0 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & 0 \\ 0 & 0 & 1 & h(\theta_1+\theta_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The set of all screw displacements about the z-axis can be described by a matrix function $\mathbf{A}(\theta)$, with the property $\mathbf{A}(\theta_1) \times \mathbf{A}(\theta_2) = \mathbf{A}(\theta_1 + \theta_2)$. Thus this one-dimensional subgroup is isomorphic to the set R^1 with the binary operation of addition. Such one-dimensional subgroups



Velocity analysis

Work out details with students on the board

Lie Algebra of the Lie group SO(3)

- Work out details with the students on the board.

In general: Definition of Lie Algebra

a *Lie algebra* is a vector space $\mathfrak{g}$ with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$; $(X, Y) \mapsto [X, Y]$ called a *Lie bracket*, such that:

(a) **Antisymmetry**: $[Y, X] = -[X, Y]$
(b) **Bilinearity**: for all $a,b \in \mathbb{F}$ and for all $X,Y,Z \in \mathfrak{g}$
  (i) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
  (ii) $[X, aY + bZ] = a[X, Y] + b[X, Z]$.
(c) **The Jacobi Identity**: for all $X,Y,Z \in \mathfrak{g}$, $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$.

What would happen if we use unit quaternions to represent rotations?

Get used to see things from different points of view, especially for rotations

Example 3.2 (Rotation of a coordinate frame and rotation of a vector) *Consider the 2D example in Figure 3.3, where on the left, a vector x is rotated clockwise by an angle α to $x_\star$. This is equivalent to (on the right) rotating the coordinate frame v counterclockwise by an angle α. Note that $x_\star^v = x^u$.*

Figure 3.3: Left: clockwise rotation α of the vector x to the vector $x_\star$. Right: counterclockwise rotation α of the coordinate frame v to the coordinate frame u.

In Figure 3.4, a vector x is rotated an angle α around the unit vector n. We denote the rotated vector by $x_\star$. Suppose that x as expressed in the coordinate frame v is known (and denoted x^{v}) and that we want to express $x_\star^{\mathrm{v}}$ in terms of x^{v}, α and n.

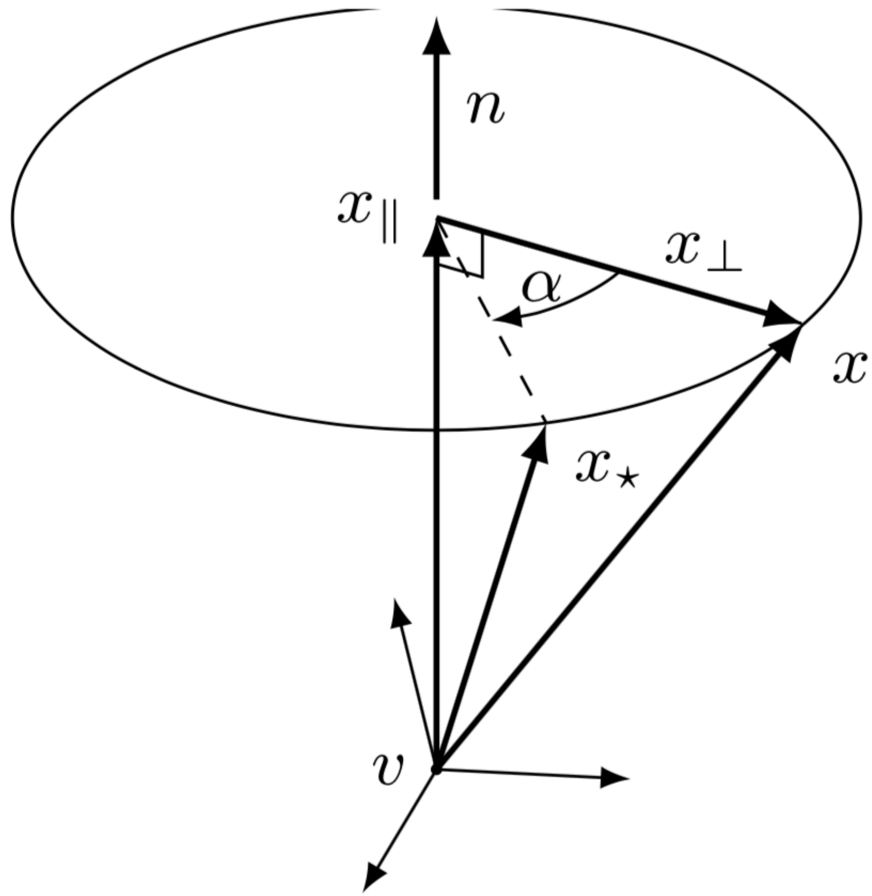


Figure 3.4: Clockwise rotation of a vector x by an angle α around the unit vector n. The rotated vector is denoted by $x_\star$. The vector x is decomposed in a component $x_{\|}$ that is parallel to the axis n, and a component $x_\perp$ that is orthogonal to it.

It can first be recognized that the vector x can
be decomposed into a component parallel to the axis n, denoted $x_{\|}$, and a component orthogonal to it, denoted $x_{\perp}$, as

$$x^{\mathrm{v}} = x_{\|}^{\mathrm{v}} + x_{\perp}^{\mathrm{v}}. \tag{3.10a}$$

Based on geometric reasoning we can conclude that

$$x_{\|}^{\mathrm{v}} = (x^{\mathrm{v}} \cdot n^{\mathrm{v}})\, n^{\mathrm{v}}, \tag{3.10b}$$

where $\cdot$ denotes the inner product. Similarly, $x_{\star}^{\mathrm{v}}$ can be decomposed as

$$x_{\star}^{\mathrm{v}} = (x_{\star}^{\mathrm{v}})_{\|} + (x_{\star}^{\mathrm{v}})_{\perp}\,, \tag{3.11a}$$

where

$$\begin{aligned} (x_{\star}^{\mathrm{v}})_{\|} &= x_{\|}^{\mathrm{v}}, & (3.11\mathrm{b}) \\ (x_{\star}^{\mathrm{v}})_{\perp} &= x_{\perp}^{\mathrm{v}} \cos\alpha + (x^{\mathrm{v}} \times n^{\mathrm{v}}) \sin\alpha. & (3.11\mathrm{c}) \end{aligned}$$

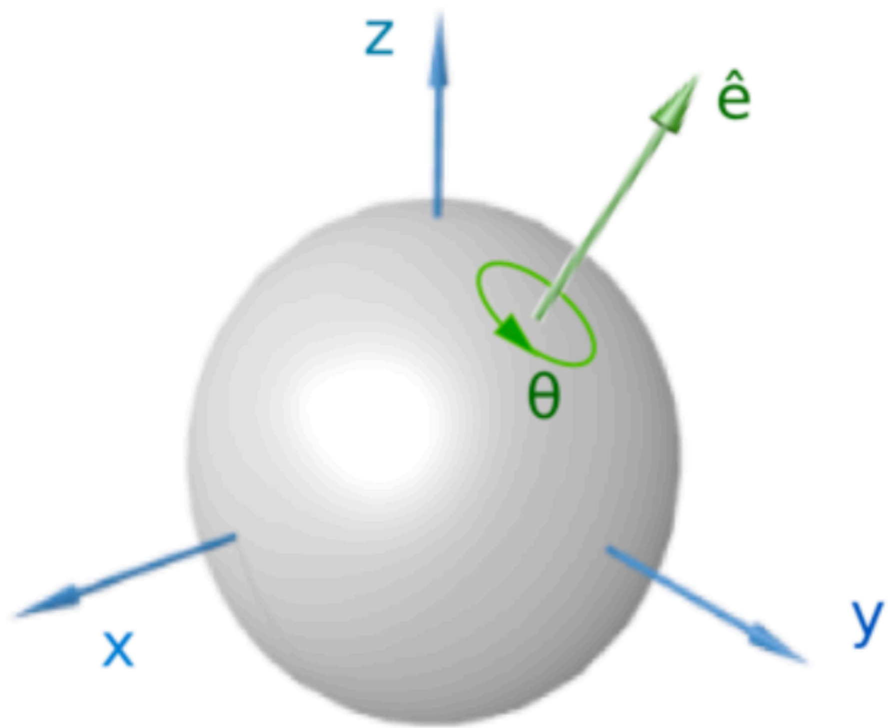
Hence, $x_{\star}^{\mathrm{v}}$ can be expressed in terms of x^{v} as

$$\begin{aligned} x_{\star}^{\mathrm{v}} &= (x^{\mathrm{v}} \cdot n^{\mathrm{v}})n^{\mathrm{v}} + (x^{\mathrm{v}} - (x^{\mathrm{v}} \cdot n^{\mathrm{v}})n^{\mathrm{v}}) \cos\alpha + (x^{\mathrm{v}} \times n^{\mathrm{v}}) \sin\alpha \\ &= x^{\mathrm{v}} \cos\alpha + n^{\mathrm{v}}(x^{\mathrm{v}} \cdot n^{\mathrm{v}})(1 - \cos\alpha) - (n^{\mathrm{v}} \times x^{\mathrm{v}}) \sin\alpha. \end{aligned} \tag{3.12}$$

Denoting the rotated coordinate frame the u-frame and using the equivalence between $x_{\star}^{\mathrm{v}}$ and x^{u} as shown in Example 3.2, this implies that

$$x^{\mathrm{u}} = x^{\mathrm{v}} \cos\alpha + n^{\mathrm{v}}(x^{\mathrm{v}} \cdot n^{\mathrm{v}})(1 - \cos\alpha) - (n^{\mathrm{v}} \times x^{\mathrm{v}}) \sin\alpha. \tag{3.13}$$

This equation is commonly referred to as the *rotation formula* or *Euler's formula* [135]. Note that the combination of n and α, or $\eta = n\alpha$, is denoted as the *rotation vector* or the *axis-angle parameterization.*



- Visualizing a rotation represented by an Euler axis and angle.

Extension of Euler's formula

A Euclidean vector such as $(2, 3, 4)$ or (a_x, a_y, a_z) can be rewritten as $2\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}$ or $a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors representing the three Cartesian axes. A rotation through an angle of θ around the axis defined by a unit vector

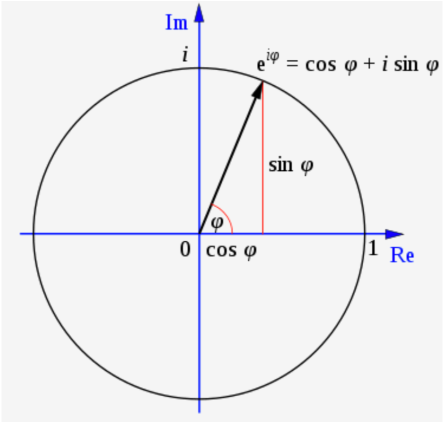
$$\vec{u} = (u_x, u_y, u_z) = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

can be represented by a quaternion. This can be done using an extension of Euler's formula:

$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})} = \cos\frac{\theta}{2} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}$$

Recall: Euler's formula:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$



Inverse and Composition

$$\mathbf{q}^{-1} = e^{-\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} - (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}.$$

It follows that conjugation by the product of two quaternions is the composition of conjugations by these quaternions: If $\mathbf{p}$ and $\mathbf{q}$ are unit quaternions, then rotation (conjugation) by $\mathbf{pq}$ is

$$\mathbf{pq}\vec{v}(\mathbf{pq})^{-1} = \mathbf{pq}\vec{v}\mathbf{q}^{-1}\mathbf{p}^{-1} = \mathbf{p}(\mathbf{q}\vec{v}\mathbf{q}^{-1})\mathbf{p}^{-1},$$

which is the same as rotating (conjugating) by $\mathbf{q}$ and then by $\mathbf{p}$. The scalar component of the result is necessarily zero.

Euler Angle

Rotation can also be defined as a consecutive rotation around three axes in terms of so-called *Euler angles*. We use the convention (z, y, x) which first rotates an angle ψ around the z-axis, subsequently an angle θ around the y-axis and finally an angle ϕ around the x-axis. These angles are illustrated in Figure 3.5. Assuming that the v-frame is rotated by (ψ, θ, ϕ) with respect to the u-frame as illustrated in this figure, the rotation matrix R^{uv} is given by

$$\begin{aligned} R^{\mathrm{uv}} &= R^{\mathrm{uv}}(e_1,\phi)R^{\mathrm{uv}}(e_2,\theta)R^{\mathrm{uv}}(e_3,\psi) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{pmatrix}, \end{aligned} \tag{3.20}$$

where we make use of the notation introduced in (3.17) and the following definition of the unit vectors

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^{\mathsf{T}}, \qquad e_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^{\mathsf{T}}, \qquad e_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^{\mathsf{T}}. \tag{3.21}$$

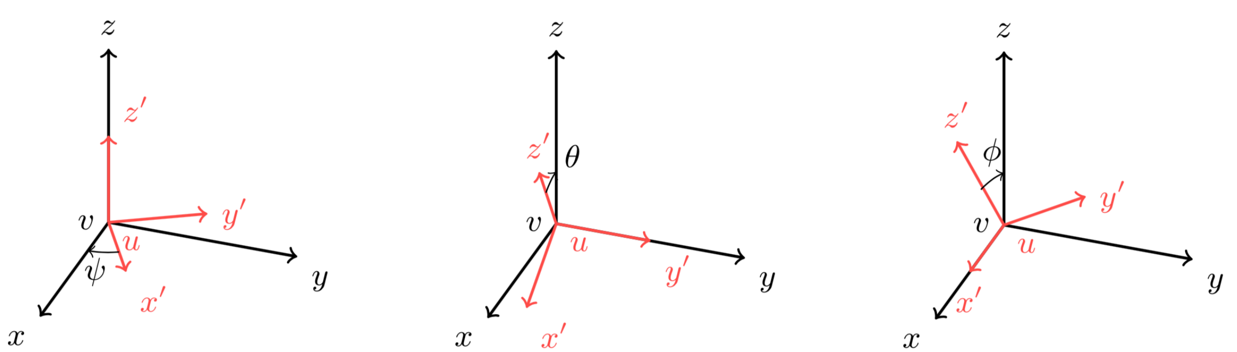


Figure 3.5: Definition of Euler angles as used in this work with left: rotation ψ around the z-axis, middle: rotation θ around the y-axis and right: rotation ϕ around the x-axis.

Pitch, roll, and yaw

The ψ, θ, ϕ angles are also often referred to as yaw (or heading), pitch and roll, respectively. Furthermore, roll and pitch together are often referred to as inclination.

Gimbal Lock

Similar to the rotation vector, Euler angles parametrize orientation as a three-dimensional vector. Euler angle representations are not unique descriptions of a rotation for two reasons. First, due to wrapping of the Euler angles, the rotation $(0,0,0)$ is for instance equal to $(0,0,2\pi k)$ for any integer k. Furthermore, setting $\theta = \frac{\pi}{2}$ in (3.20), leads to

$$
\begin{aligned}
R^{\mathrm{uv}} &= \begin{pmatrix} 0 & 0 & -1 \\ \sin\phi\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ \cos\phi\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\psi - \sin\phi\cos\psi & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & -1 \\ \sin(\phi-\psi) & \cos(\phi-\psi) & 0 \\ \cos(\phi-\psi) & -\sin(\phi-\psi) & 0 \end{pmatrix}.
\end{aligned} \tag{3.22}
$$

Hence, only the rotation $\phi - \psi$ can be observed. Because of this, for example the rotations $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2}, -\frac{\pi}{2})$, $(\pi, \frac{\pi}{2}, \frac{\pi}{2})$ are all three equivalent. This is called *gimbal lock* [31].

- We will analyze further using quaternions next.

Exponential an skew symmetric matrix, we get a rotation matrix

Recall:

Given a square matrix $X \in \mathbb{R}^{n\times n}$,
the exponential of X is given by the absolute convergent power series

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

What is a rigid motion/transformation?

A rigid transformation is formally defined as a transformation that, when acting on any vector **v**, produces a transformed vector $T(\mathbf{v})$ of the form

$$T(\mathbf{v}) = R\,\mathbf{v} + \mathbf{t}$$

where $R^{\mathsf{T}} = R^{-1}$ (i.e., R is an orthogonal transformation), and **t** is a vector giving the translation of the origin.

A proper rigid transformation has, in addition,

$$\det(\mathrm{R}) = 1$$

which means that R does not produce a reflection, and hence it represents a rotation (an orientation-preserving orthogonal transformation). Indeed, when an orthogonal transformation matrix produces a reflection, its determinant is -1.

Note: Rotation and Translation do not commute

$$T(\mathbf{v}) = R\,\mathbf{v} + \mathbf{t}$$

- This can be viewed as first rotating the vector v by using rotation matrix R, then translate by t.
- What if we first translate the vector v by t and then do the rotation by R?

Key property of any rigid motion: preserves the distance!

$$d(g(\mathbf{X}), g(\mathbf{Y}))^2 = d(\mathbf{X}, \mathbf{Y})^2.$$

$$\begin{aligned} d(\mathbf{X}, \mathbf{Y})^2 &= (X_1 - Y_1)^2 + (X_2 - Y_2)^2 + \ldots + (X_n - Y_n)^2 \\ &= (\mathbf{X} - \mathbf{Y}) \cdot (\mathbf{X} - \mathbf{Y}). \end{aligned}$$

For the Euclidean distance d and

a rigid transformation g:$\mathrm{R}^n \rightarrow \mathrm{R}^n$

Claim: The tangent plane of the Lie group SO(3) at the identity is the set of skew-symmetric matrices.

Work out details with the students on the board.

Even we are interested in n=3. All the derivations can be extended to dim=n.

$$SO(n) = \{Q \in \mathbb{R}^{n\times n} : \; Q^\top = Q^{-1}, \; \det(Q) = 1\}.$$

The algebra of skew-symmetric matrices is denoted by

$$so(n) = \{S \in \mathbb{R}^{n\times n} : \; S^\top = -S\}$$

so(n) is called the Lie algebra of the Lie group SO(n).

The set of rigid motions SE(n) also form a Lie Group

$$SE(n) = \left\{ \begin{bmatrix} Q & \mathbf{u} \\ \mathbf{0} & 1 \end{bmatrix} : Q \in SO(n), \mathbf{u} \in \mathbb{R}^{n\cancel{\times 1}} \right\}.$$

While $SE(n)$ describes configurations, its Lie algebra $se(n)$, defined by

$$se(n) = \left\{ \begin{bmatrix} S & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} : S \in so(n), \mathbf{v} \in \mathbb{R}^{n\cancel{\times 1}} \right\},$$

- Note: Here we used the homogeneous representation of rotation and translation.

Geometric aspect of the exponential and logarithm

To any vector $\mathbf{u} = (u_1,\ u_2,\ u_3) \in \mathbb{R}^3$ one can associate the 3×3 skew-symmetric matrix

$$S_{\mathbf{u}} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

It is easy to see that, for every $\mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \times \mathbf{v} = S_{\mathbf{u}}\, \mathbf{v},$$

where $\times$ stands for the cross product.

Now consider the following problem: given a unit vector $\mathbf{u} \in \mathbb{R}^3$ and an angle $\theta \in \mathbb{R}$, find the rotation matrix R that rotates any vector through the angle θ about an axis with direction $\mathbf{u}$.

The matrix exponential gives the elegant solution:

$$R = e^{S_{\mathbf{u}}\theta}.$$

- Moreover, we have a closed formula for it:

$$e^{S_{\mathbf{u}}\theta} = I + \sin\theta\, S_{\mathbf{u}} + (1 - \cos\theta)\, S_{\mathbf{u}}^2.$$

- The above is the famous **Rodrigues Formula.**

Derivation of Rodrigues Formula

- Work out the details with the students.

Derivation of Rodrigues' Rotation Formula

In the theory of three-dimensional rotation, **Rodrigues' rotation formula**, named after Olinde Rodrigues, is an efficient algorithm for rotating a vector in space, given an axis and angle of rotation. By extension, this can be used to transform all three basis vectors to compute a rotation matrix in $\mathrm{SO}(3)$, the group of all rotation matrices, from an axis–angle representation. In other words, the Rodrigues' formula provides an algorithm to compute the exponential map from $\mathfrak{so}(3)$, the Lie algebra of $\mathrm{SO}(3)$, to $\mathrm{SO}(3)$ without actually computing the full matrix exponential.

Key idea: Only rotate the Perpendicular part

Let $\mathbf{k}$ be a unit vector defining a rotation axis, and let $\mathbf{v}$ be any vector to rotate about $\mathbf{k}$ by angle θ (right hand rule, anticlockwise in the figure).

Using the dot and cross products, the vector $\mathbf{v}$ can be decomposed into components parallel and perpendicular to the axis $\mathbf{k}$,

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \,,$$

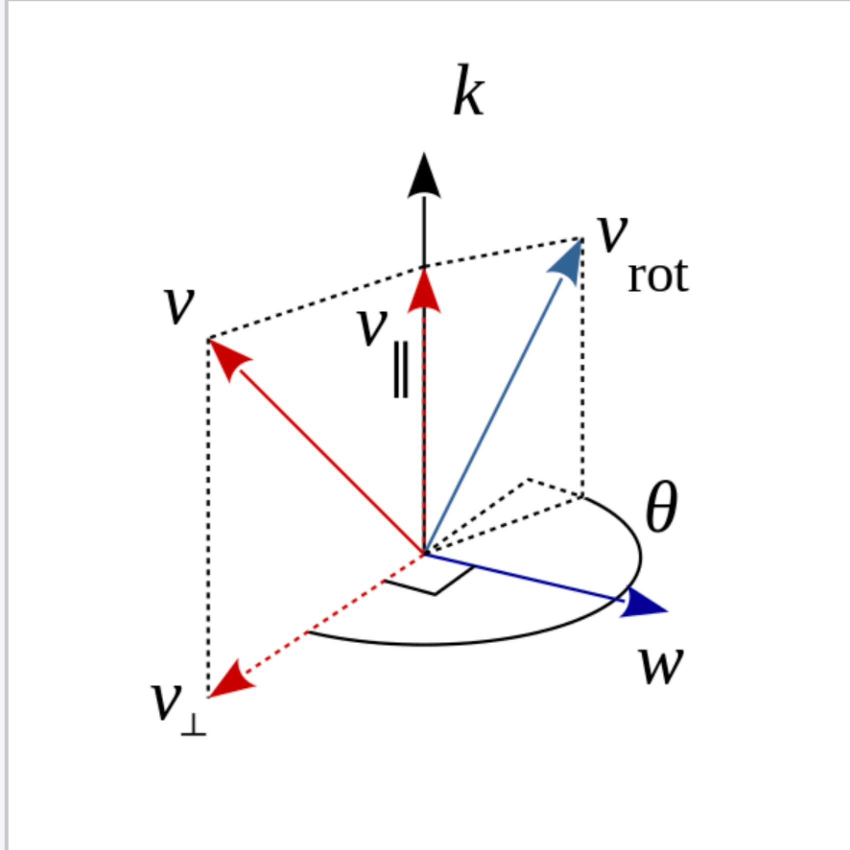
where the component parallel to $\mathbf{k}$ is

$$\mathbf{v}_{\parallel} = (\mathbf{v} \cdot \mathbf{k})\mathbf{k}$$

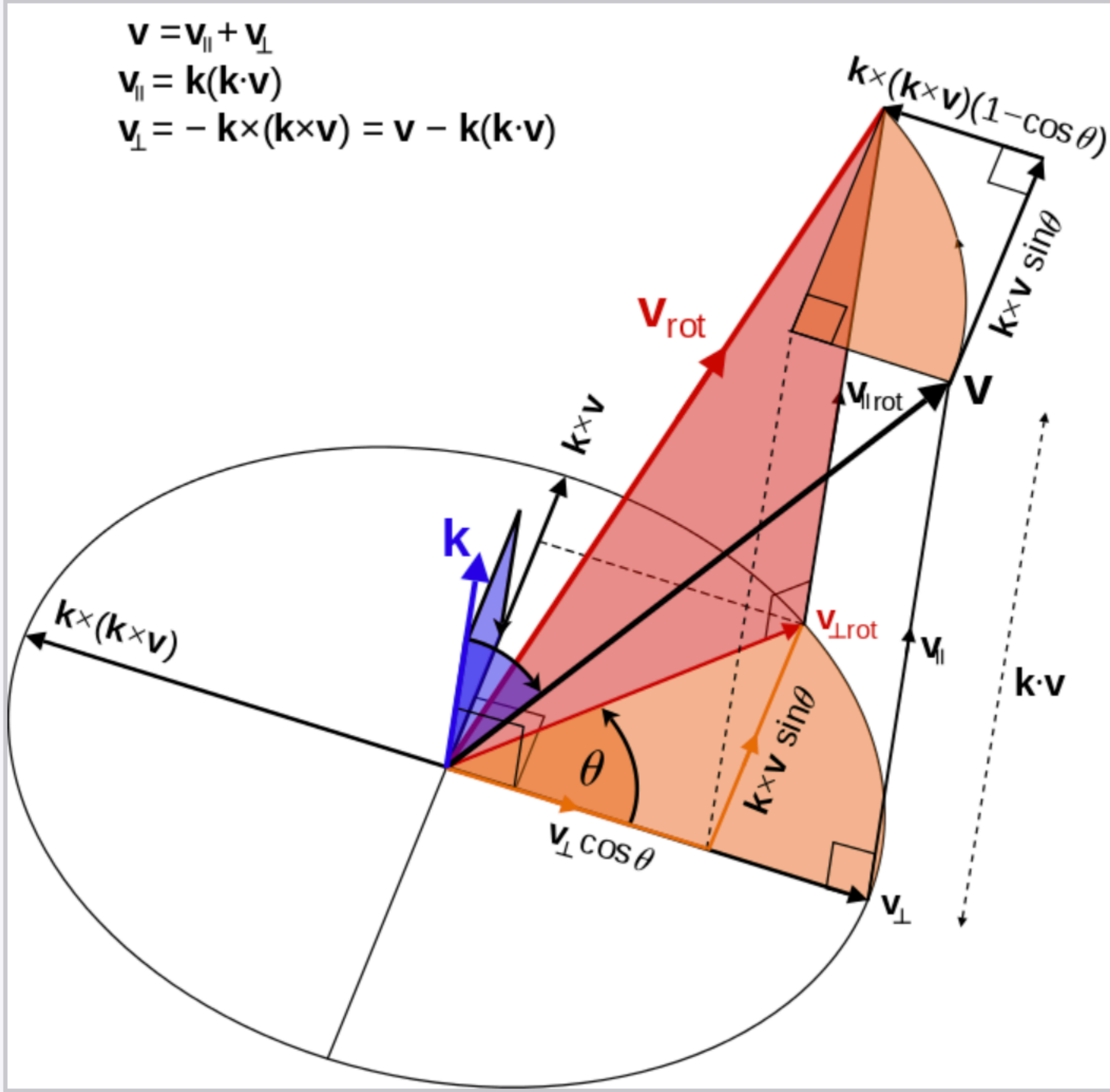
called the vector projection of $\mathbf{v}$ on $\mathbf{k}$, and the component perpendicular to $\mathbf{k}$ is

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = \mathbf{v} - (\mathbf{k} \cdot \mathbf{v})\mathbf{k} = -\mathbf{k} \times (\mathbf{k} \times \mathbf{v})$$

called the vector rejection of $\mathbf{v}$ from $\mathbf{k}$.



Rodrigues' rotation formula rotates $\mathbf{v}$ by an angle θ around vector k by decomposing it into its components parallel and perpendicular to k, and rotating only the perpendicular component.



Vector geometry of Rodrigues' rotation formula, as well as the decomposition into parallel and perpendicular components.

The vector $\mathbf{k} \times \mathbf{v}$ can be viewed as a copy of $\mathbf{v}_\perp$ rotated anticlockwise by 90° about $\mathbf{k}$, so their magnitudes are equal but directions are perpendicular. Likewise the vector $\mathbf{k} \times (\mathbf{k} \times \mathbf{v})$ a copy of $\mathbf{v}_\perp$ rotated anticlockwise through $180°$ about $\mathbf{k}$, so that $\mathbf{k} \times (\mathbf{k} \times \mathbf{v})$ and $\mathbf{v}_\perp$ are equal in magnitude but in opposite directions (i.e. they are negatives of each other, hence the minus sign). Expanding the vector triple product establishes the connection between the parallel and perpendicular components, for reference the formula is $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ given any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

The component parallel to the axis will not change magnitude nor direction under the rotation,

$$\mathbf{v}_{\parallel \mathrm{rot}} = \mathbf{v}_\parallel \ ,$$

only the perpendicular component will change direction but retain its magnitude, according to

$$\begin{aligned} |\mathbf{v}_{\perp \mathrm{rot}}| &= |\mathbf{v}_\perp| \ , \\ \mathbf{v}_{\perp \mathrm{rot}} &= \cos\theta \mathbf{v}_\perp + \sin\theta \mathbf{k} \times \mathbf{v}_\perp \ , \end{aligned}$$

and since $\mathbf{k}$ and $\mathbf{v}_\parallel$ are parallel, their cross product is zero $\mathbf{k} \times \mathbf{v}_\parallel = \mathbf{0}$, so that

$$\mathbf{k} \times \mathbf{v}_\perp = \mathbf{k} \times \left(\mathbf{v} - \mathbf{v}_\parallel\right) = \mathbf{k} \times \mathbf{v} - \mathbf{k} \times \mathbf{v}_\parallel = \mathbf{k} \times \mathbf{v}$$

and it follows

$$\mathbf{v}_{\perp\mathrm{rot}} = \cos\theta \mathbf{v}_\perp + \sin\theta \mathbf{k} \times \mathbf{v}\,.$$

This rotation is correct since the vectors $\mathbf{v}_\perp$ and $\mathbf{k} \times \mathbf{v}$ have the same length, and $\mathbf{k} \times \mathbf{v}$ is $\mathbf{v}_\perp$ rotated anticlockwise through $90°$ about $\mathbf{k}$. An appropriate scaling of $\mathbf{v}_\perp$ and $\mathbf{k} \times \mathbf{v}$ using the trigonometric functions sine and cosine gives the rotated perpendicular component. The form of the rotated component is similar to the radial vector in 2D planar polar coordinates (r, θ) in the Cartesian basis

$$\mathbf{r} = r\cos\theta \mathbf{e}_x + r\sin\theta \mathbf{e}_y\,,$$

where $\mathbf{e}_x$, $\mathbf{e}_y$ are unit vectors in their indicated directions.

Now the full rotated vector is

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{v}_{\parallel\mathrm{rot}} + \mathbf{v}_{\perp\mathrm{rot}}\,,$$

Now the full rotated vector is

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{v}_{\parallel\mathrm{rot}} + \mathbf{v}_{\perp\mathrm{rot}} \ ,$$

By substituting the definitions of $\mathbf{v}_{\parallel\mathrm{rot}}$ and $\mathbf{v}_{\perp\mathrm{rot}}$ in the equation results in

$$\begin{aligned}
\mathbf{v}_{\mathrm{rot}} &= \mathbf{v}_{\parallel} + \cos\theta\,\mathbf{v}_{\perp} + \sin\theta\,\mathbf{k}\times\mathbf{v} \\
&= \mathbf{v}_{\parallel} + \cos\theta\left(\mathbf{v} - \mathbf{v}_{\parallel}\right) + \sin\theta\,\mathbf{k}\times\mathbf{v} \\
&= \cos\theta\,\mathbf{v} + (1-\cos\theta)\mathbf{v}_{\parallel} + \sin\theta\,\mathbf{k}\times\mathbf{v} \\
&= \cos\theta\,\mathbf{v} + (1-\cos\theta)(\mathbf{k}\cdot\mathbf{v})\mathbf{k} + \sin\theta\,\mathbf{k}\times\mathbf{v}
\end{aligned}$$

Moreover, since $\mathbf{k}$ is a unit vector, $\mathbf{K}$ has unit 2-norm. The previous rotation formula in matrix language is therefore

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{v} + (\sin\theta)\mathbf{K}\mathbf{v} + (1-\cos\theta)\mathbf{K}^2\mathbf{v}\,, \quad \|\mathbf{K}\|_2 = 1\,.$$

Note the coefficient of the leading term is *now* 1, in this notation: see the Lie-Group discussion below.

Factorizing the $\mathbf{v}$ allows the compact expression

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{R}\mathbf{v}$$

where

$$\boxed{\mathbf{R} = \mathbf{I} + (\sin\theta)\mathbf{K} + (1-\cos\theta)\mathbf{K}^2}$$

is the rotation matrix through an angle θ counterclockwise about the axis $\mathbf{k}$, and $\mathbf{I}$ the 3 × 3 identity matrix. This matrix $\mathbf{R}$ is an element of the rotation group $\mathrm{SO}(3)$ of $\mathbb{R}^3$, and $\mathbf{K}$ is an element of the Lie algebra $\mathfrak{so}(3)$ generating that Lie group (note that $\mathbf{K}$ is skew-symmetric, which characterizes $\mathfrak{so}(3)$).

In terms of the matrix exponential,

$$\mathbf{R} = \exp(\theta\mathbf{K}) \,.$$

To see that the last identity holds, one notes that

$$\mathbf{R}(\theta)\mathbf{R}(\phi) = \mathbf{R}(\theta + \phi), \quad \mathbf{R}(0) = \mathbf{I} \,,$$

characteristic of a one-parameter subgroup, i.e. exponential, and that the formulas match for infinitesimal θ.

For an alternative derivation based on this exponential relationship, see exponential map from $\mathbf{so}(3)$ to $\mathrm{SO}(3)$. For the inverse mapping, see log map from $\mathrm{SO}(3)$ to $\mathbf{so}(3)$.

Note that the Hodge dual of the rotation $\mathbf{R}$ is just $\mathbf{R}^* = -\sin(\theta)\mathbf{k}$ which allows the extraction of both the axis of rotation and the sine of the angle of the rotation from the rotation itself, with the usual ambiguity:

$$\sin(\theta) = \sigma|\mathbf{R}^*|$$

$$\mathbf{k} = -\sigma\mathbf{R}^*/|\mathbf{R}^*|$$

where $\sigma = \pm 1$. The above simple expression results from the fact that the Hodge dual of $\mathbf{I}$ and $\mathbf{K}^2$ are zero, and $\mathbf{K}^* = -\mathbf{k}$.

Representing $\mathbf{v}$ and $\mathbf{k} \times \mathbf{v}$ as column matrices, the cross product can be expressed as a matrix product

$$\begin{bmatrix} (\mathbf{k} \times \mathbf{v})_x \\ (\mathbf{k} \times \mathbf{v})_y \\ (\mathbf{k} \times \mathbf{v})_z \end{bmatrix} = \begin{bmatrix} k_y v_z - k_z v_y \\ k_z v_x - k_x v_z \\ k_x v_y - k_y v_x \end{bmatrix} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}.$$

Letting $\mathbf{K}$ denote the "cross-product matrix" for the unit vector $\mathbf{k}$,

$$\mathbf{K} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix},$$

Matrix Rotation

the matrix equation is, symbolically,

$$\mathbf{K}\mathbf{v} = \mathbf{k} \times \mathbf{v}$$

for any vector $\mathbf{v}$. (In fact, $\mathbf{K}$ is the unique matrix with this property. It has eigenvalues 0 and $\pm i$).

Iterating the cross product on the right is equivalent to multiplying by the cross product matrix on the left, in particular

$$\mathbf{K}(\mathbf{K}\mathbf{v}) = \mathbf{K}^2\mathbf{v} = \mathbf{k} \times (\mathbf{k} \times \mathbf{v}).$$

Exponential map from $\mathfrak{so}(3)$ to SO(3)

The exponential map effects a transformation from the axis-angle representation of rotations to rotation matrices,

$$\exp\colon \mathfrak{so}(3) \to \mathrm{SO}(3)\,.$$

Essentially, by using a Taylor expansion one derives a closed-form relation between these two representations. Given a unit vector $\boldsymbol{\omega} \in \mathfrak{so}(3) = \mathbb{R}^3$ representing the unit rotation axis, and an angle, $\theta \in \mathbb{R}$, an equivalent rotation matrix R is given as follows, where $\mathbf{K}$ is the cross product matrix of $\boldsymbol{\omega}$, that is, $\mathbf{K}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$ for all vectors $\mathbf{v} \in \mathbb{R}^3$,

$$R = \exp(\theta\mathbf{K}) = \sum_{k=0}^{\infty} \frac{(\theta\mathbf{K})^k}{k!} = I + \theta\mathbf{K} + \frac{1}{2!}(\theta\mathbf{K})^2 + \frac{1}{3!}(\theta\mathbf{K})^3 + \cdots$$

Lie-Algebraic derivation of Rodrigues' Rotation Formula

Because $\mathbf{K}$ is skew-symmetric, and the sum of the squares of its above-diagonal entries is 1, the characteristic polynomial $P(t)$ of $\mathbf{K}$ is $P(t) = \det(\mathbf{K} - t\mathbf{I}) = -(t^3 + t)$. Since, by the Cayley–Hamilton theorem, $P(\mathbf{K}) = 0$, this implies that

$$\mathbf{K}^3 = -\mathbf{K}\,.$$

As a result, $\mathbf{K}^4 = -\mathbf{K}^2$, $\mathbf{K}^5 = \mathbf{K}$, $\mathbf{K}^6 = \mathbf{K}^2$, $\mathbf{K}^7 = -\mathbf{K}$.

This cyclic pattern continues indefinitely, and so all higher powers of $\mathbf{K}$ can be expressed in terms of $\mathbf{K}$ and $\mathbf{K}^2$. Thus, from the above equation, it follows that

$$R = I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\mathbf{K} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)\mathbf{K}^2\,,$$

that is,

$$R = I + (\sin\theta)\mathbf{K} + (1 - \cos\theta)\mathbf{K}^2\,.$$

This is a Lie-algebraic derivation, in contrast to the geometric one in the article Rodrigues' rotation formula.[1]

Log map from SO(3) to $\mathfrak{so}(3)$

Let $\mathbf{K}$ continue to denote the 3 × 3 matrix that effects the cross product with the rotation axis $\boldsymbol{\omega}$: $\mathbf{K}(\mathbf{v}) = \boldsymbol{\omega} \times \mathbf{v}$ for all vectors $\mathbf{v}$ in what follows.

To retrieve the axis–angle representation of a rotation matrix, calculate the angle of rotation from the trace of the rotation matrix

$$\theta = \arccos\left(\frac{\operatorname{Tr}(R) - 1}{2}\right)$$

and then use that to find the normalized axis,

$$\boldsymbol{\omega} = \frac{1}{2\sin\theta}\begin{bmatrix} R(3,2) - R(2,3) \\ R(1,3) - R(3,1) \\ R(2,1) - R(1,2) \end{bmatrix}.$$

Note also that the Matrix logarithm of the rotation matrix R is

$$\log R = \begin{cases} 0 & \text{if } \theta = 0 \\ \dfrac{\theta}{2\sin\theta}\left(R - R^{\mathsf{T}}\right) & \text{if } \theta \neq 0 \text{ and } \theta \in (-\pi, \pi) \end{cases}$$

An exception occurs when R has eigenvalues equal to –1. In this case, the log is not unique. However, even in the case where $\theta = \pi$ the Frobenius norm of the log is

$$\|\log(R)\|_{\mathrm{F}} = \sqrt{2}|\theta|\,.$$

Logarithm of a Matrix

The exponential of a matrix *A* is defined by

$$e^A \equiv \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Given a matrix *B*, another matrix *A* is said to be a **matrix logarithm** of B if $e^A = B$. Because the exponential function is not one-to-one for complex numbers (e.g. $e^{\pi i} = e^{3\pi i} = -1$), numbers can have multiple complex logarithms, and as a consequence of this, some matrices may have more than one logarithm, as explained below.

If *B* is sufficiently close to the identity matrix, then a logarithm of *B* may be computed by means of the following power series:

$$\log(B) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(B-I)^k}{k} = (B-I) - \frac{(B-I)^2}{2} + \frac{(B-I)^3}{3} \cdots.$$

Specifically, if $\|B - I\| < 1$, then the preceding series converges and $e^{\log(B)} = B$.[1]

Example:
Logarithm of rotations in the plane

The rotations in the plane give a simple example. A rotation of angle α around the origin is represented by the 2×2-matrix

$$A = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

For any integer n, the matrix

$$B_n = (\alpha + 2\pi n)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is a logarithm of A. Thus, the matrix A has infinitely many logarithms. This corresponds to the fact that the rotation angle is only determined up to multiples of 2π.

In the language of Lie theory, the rotation matrices A are elements of the Lie group SO(2). The corresponding logarithms B are elements of the Lie algebra so(2), which consists of all skew-symmetric matrices. The matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a generator of the Lie algebra so(2).

Logarithm of rotations in 3D space

A rotation $R \in SO(3)$ in $\mathbb{R}^3$ is given by a 3×3 orthogonal matrix.

The logarithm of such a rotation matrix R can be readily computed from the antisymmetric part of Rodrigues' rotation formula[5] (see also Axis angle). It yields the logarithm of minimal Frobenius norm, but fails when R has eigenvalues equal to −1 where this is not unique.

Further note that, given rotation matrices *A* and *B*,

$$d_g(A, B) := \| \log(A^\top B) \|_F$$

is the geodesic distance on the 3D manifold of rotation matrices.

Calculating the logarithm of a diagonalizable matrix

A method for finding ln A for a diagonalizable matrix A is the following:

Find the matrix V of eigenvectors of A (each column of V is an eigenvector of A).

Find the inverse V^{-1} of V.

Let

$$A' = V^{-1}AV.$$

Then A' will be a diagonal matrix whose diagonal elements are eigenvalues of A.

Replace each diagonal element of A' by its (natural) logarithm in order to obtain $\log A'$.

Then

$$\log A = V(\log A')V^{-1}.$$

That the logarithm of A might be a complex matrix even if A is real then follows from the fact that a matrix with real and positive entries might nevertheless have negative or even complex eigenvalues (this is true for example for rotation matrices). The non-uniqueness of the logarithm of a matrix follows from the non-uniqueness of the logarithm of a complex number.

The logarithm of a non-diagonalizable matrix

The algorithm illustrated above does not work for non-diagonalizable matrices, such as

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

For such matrices one needs to find its Jordan decomposition and, rather than computing the logarithm of diagonal entries as above, one would calculate the logarithm of the Jordan blocks.

The latter is accomplished by noticing that one can write a Jordan block as

$$B = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 & \lambda^{-1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & \lambda^{-1} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \lambda^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & \lambda^{-1} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \lambda(I + K)$$

where K is a matrix with zeros on and under the main diagonal. (The number λ is nonzero by the assumption that the matrix whose logarithm one attempts to take is invertible.)

Then, by the Mercator series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

one gets

$$\log B = \log\big(\lambda(I+K)\big) = \log(\lambda I) + \log(I+K) = (\log\lambda)I + K - \frac{K^2}{2} + \frac{K^3}{3} - \frac{K^4}{4} + \cdots$$

This series has a finite number of terms (K^m is zero if m is the dimension of K), and so its sum is well-defined.

Using this approach one finds

$$\log\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

A Lie group theory perspective

In the theory of Lie groups, there is an exponential map from a Lie algebra *g* to the corresponding Lie group *G*

$$\exp : g \to G.$$

For matrix Lie groups, the elements of *g* and *G* are square matrices and the exponential map is given by the matrix exponential. The inverse map $\log = \exp^{-1}$ is multivalued and coincides with the matrix logarithm discussed here. The logarithm maps from the Lie group *G* into the Lie algebra *g*. Note that the exponential map is a local diffeomorphism between a neighborhood *U* of the zero matrix $\underline{0} \in g$ and a neighborhood *V* of the identity matrix $\underline{1} \in G$.[6] Thus the (matrix) logarithm is well-defined as a map,

$$\log : V \subset G \to U \subset g.$$

An important corollary of Jacobi's formula then is

$$\log(\det(A)) = \mathrm{tr}(\log A) \ .$$

Properties

If A and B are both positive-definite matrices, then

$$\operatorname{tr}\log(AB) = \operatorname{tr}\log(A) + \operatorname{tr}\log(B),$$

and if A and B commute, i.e., $AB = BA$, then

$$\log(AB) = \log(A) + \log(B).$$

Substituting in this equation $B = A^{-1}$, one gets

$$\log(A^{-1}) = -\log(A).$$

The power of Exponential and Log of matrices!

Conversely, given $R \in SO(3)$ (with no negative eigenvalues) consider the problem of finding the axis direction $\mathbf{u}$ and the angle θ of rotation. Using the matrix exponential,
we can formulate this problem as follows:
determine a unitary vector $\mathbf{u}$ and $\theta \in]-\pi, \pi[$ such that

$$R = e^{S_{\mathbf{u}}\theta}.$$

The matrix logarithm provides the simple answer

$$S_{\mathbf{u}}\theta = \log(R),$$

or equivalently,

$$S_{\mathbf{u}} = \frac{1}{\theta}\log(R), \quad \text{whenever } \theta \neq 0.$$

A widely used formula for computing the logarithm of a 3×3 rotation matrix is

$$\log R = \frac{\theta}{2\sin\theta}(R^\top - R),$$

where θ satisfies $1 + 2\cos\theta = \mathrm{trace}(R), \theta \neq 0, -\pi < \theta < \pi$.

When $\theta = 0$ one has the trivial case $R = I$ and $\log R = 0$.

- Note:

How to calculate the angle?

- Taking the norm both sides of

$$S_{\mathbf{u}} = \frac{1}{\theta} \log(R),$$

- We get

$$\| S_{\mathbf{u}} \| \, |\theta| = \| \log(R) \|,$$

and, since $\| S_{\mathbf{u}} \| = 1$, the angle of rotation is related with the norm of $\log(R)$ by

$$|\theta| = \| \log(R) \|.$$

Summarize

- This means that the direction of the rotation axis is given by the logarithm of the matrix associated with u.

The power of Exponential and Log of matrices!

Conversely, given $R \in SO(3)$ (with no negative eigenvalues) consider the problem of finding the axis direction $\mathbf{u}$ and the angle θ of rotation. Using the matrix exponential,
we can formulate this problem as follows:
determine a unitary vector $\mathbf{u}$ and $\theta \in]-\pi, \pi[$ such that

$$R = e^{S_{\mathbf{u}}\theta}.$$

The matrix logarithm provides the simple answer

$$S_{\mathbf{u}}\theta = \log(R),$$

or equivalently,

$$S_{\mathbf{u}} = \frac{1}{\theta}\log(R), \quad \text{whenever } \theta \neq 0.$$

A widely used formula for computing the logarithm of a 3×3 rotation matrix is

$$\log R = \frac{\theta}{2\sin\theta}(R^\top - R),$$

where θ satisfies $1 + 2\cos\theta = \mathrm{trace}(R), \theta \neq 0, -\pi < \theta < \pi$.

When $\theta = 0$ one has the trivial case $R = I$ and $\log R = 0$.

- Note:

How to calculate the angle?

- Taking the norm both sides of

$$S_{\mathbf{u}} = \frac{1}{\theta} \log(R),$$

- We get

$$\| S_{\mathbf{u}} \| \, |\theta| = \| \log(R) \|,$$

and, since $\| S_{\mathbf{u}} \| = 1$, the angle of rotation is related with the norm of $\log(R)$ by

$$|\theta| = \| \log(R) \|.$$

Summarize

- This means that the direction of the rotation axis is given by the logarithm of the matrix associated with u.

- This relationship between skew-symmetric and rotation matrices by means of exponentials and logarithms are the key to explain the importance of these matrix functions in rigid motions and robotics. There are many other geometric problems involving exponentials and logarithms of matrices. For example, my student Tum and I was working on the design of trajectories for UAVs in an indoor environments (meaning assuming without GPS).

Claim: Given a unit quaternion, we can define a rotation.

What is the matrix for R_q?

- Ans: It is the matrix representation of $\mathbf{R_q}$.
- How to find it?
- Ans: Let $\mathbf{R_q}$ acts on the basis {1, i, j, k}

$$\mathbf{R} = \begin{bmatrix} 1 - 2s(q_j^2 + q_k^2) & 2s(q_i q_j - q_k q_r) & 2s(q_i q_k + q_j q_r) \\ 2s(q_i q_j + q_k q_r) & 1 - 2s(q_i^2 + q_k^2) & 2s(q_j q_k - q_i q_r) \\ 2s(q_i q_k - q_j q_r) & 2s(q_j q_k + q_i q_r) & 1 - 2s(q_i^2 + q_j^2) \end{bmatrix}$$

Here $s = ||q||^{-2}$ and if q is a unit quaternion, $s = 1$.

Recovering the axis-angle representation

The axis a and angle θ corresponding to a quaternion $\mathbf{q} = q_r + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$ can be extracted via:

$$(a_x, a_y, a_z) = \frac{(q_i, q_j, q_k)}{\sqrt{q_i^2 + q_j^2 + q_k^2}}$$

$$\theta = 2\,\mathrm{atan2}\left(\sqrt{q_i^2 + q_j^2 + q_k^2}, q_r\right),$$

where $\mathrm{atan2}$ is the two-argument arctangent. Care should be taken when the quaternion approaches a real quaternion, since due to degeneracy the axis of an identity rotation is not well-defined.

Differentiation with respect to the rotation quaternion

The rotated quaternion $\mathbf{p'}=\mathbf{q}\,\mathbf{p}\,\mathbf{q}^*$ needs to be differentiated with respect to the rotating quaternion $\mathbf{q}$, when the rotation is estimated from numerical optimization. The estimation of rotation angle is an essential procedure in 3D object registration or camera calibration. The derivative can be represented using the Matrix Calculus notation.

$$\frac{\partial \mathbf{p}'}{\partial \mathbf{q}} \equiv \left[\frac{\partial \mathbf{p}'}{\partial q_0}, \frac{\partial \mathbf{p}'}{\partial q_x}, \frac{\partial \mathbf{p}'}{\partial q_y}, \frac{\partial \mathbf{p}'}{\partial q_z}\right]$$

$$= \left[\mathbf{pq} - (\mathbf{pq})^*, (\mathbf{pqi})^* - \mathbf{pqi}, (\mathbf{pqj})^* - \mathbf{pqj}, (\mathbf{pqk})^* - \mathbf{pqk}\right].$$

Types of Matrix Derivatives

Types	Scalar	Vector	Matrix
Scalar	$\frac{\partial y}{\partial x}$	$\frac{\partial \mathbf{y}}{\partial x}$	$\frac{\partial \mathbf{Y}}{\partial x}$
Vector	$\frac{\partial y}{\partial \mathbf{x}}$	$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$	
Matrix	$\frac{\partial y}{\partial \mathbf{X}}$		

Will be covered in Math 173, Advanced linear Algebra.

We also can add a scalar to a vector and find inverse of a vector!

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a + \vec{v}.$$

$$a + \vec{v} = (a, \vec{0}) + (0, \vec{v}).$$

$$(s + \vec{v})^{-1} = \frac{(s + \vec{v})^*}{\|s + \vec{v}\|^2} = \frac{s - \vec{v}}{s^2 + \|\vec{v}\|^2}$$

Now we can multiply two vectors in R^3 and in R^4 !
First define it in R^3
by viewing them as pure imaginary quaternions

We can express quaternion multiplication in the modern language of vector cross and dot products (which were actually inspired by the quaternions in the first place [6]). When multiplying the vector/imaginary parts, in place of the rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ we have the quaternion multiplication rule:

$$\vec{v}\vec{w} = \vec{v} \times \vec{w} - \vec{v} \cdot \vec{w},$$

where:

- $\vec{v}\vec{w}$ is the resulting quaternion,
- $\vec{v} \times \vec{w}$ is vector cross product (a vector),
- $\vec{v} \cdot \vec{w}$ is vector scalar product (a scalar).

Quaternion multiplication is noncommutative (because of the cross product, which anti-commutes), while scalar–scalar and scalar–vector multiplications commute. From these rules it follows immediately that (see details):

$$(s + \vec{v})(t + \vec{w}) = (st - \vec{v} \cdot \vec{w}) + (s\vec{w} + t\vec{v} + \vec{v} \times \vec{w}).$$

Quaternions are extension of complex numbers

The complex numbers can be defined by introducing an abstract symbol $\mathbf{i}$ which satisfies the usual rules of algebra and additionally the rule $\mathbf{i}^2 = -1$. This is sufficient to reproduce all of the rules of complex number arithmetic: for example:

$$(a + b\mathbf{i})(c + d\mathbf{i}) = ac + ad\mathbf{i} + b\mathbf{i}c + b\mathbf{i}d\mathbf{i}$$
$$= ac + ad\mathbf{i} + bc\mathbf{i} + bd\mathbf{i}^2 = (ac - bd) + (bc + ad)\mathbf{i}.$$

Multiply two quaternions:

$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(e + f\mathbf{i} + g\mathbf{j} + h\mathbf{k}) =$$
$$(ae - bf - cg - dh) + (af + be + ch - dg)\mathbf{i} + (ag - bh + ce + df)\mathbf{j} + (ah + bg - cf + de)\mathbf{k}.$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1$$

Notations for using multiple frames

- Pay attention to the subscripts!

$$q = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} q_0 \\ q_v \end{pmatrix}, \qquad q \in \mathbb{R}^4, \qquad \|q\|_2 = 1.$$

A rotation can be defined using unit quaternions as

$$\bar{x}^{\mathrm{u}} = q^{\mathrm{uv}} \odot \bar{x}^{\mathrm{v}} \odot (q^{\mathrm{uv}})^{\mathsf{c}},$$

where $\cdot^{\mathsf{c}}$ denotes the quaternion conjugate, defined as

$$q^{\mathsf{c}} = \begin{pmatrix} q_0 & -q_v^{\mathsf{T}} \end{pmatrix}^{\mathsf{T}},$$

and $\bar{x}^{\mathrm{v}}$ denotes the quaternion representation of x^{v} as

$$\bar{x}^{\mathrm{v}} = \begin{pmatrix} 0 & (x^{\mathrm{v}})^{\mathsf{T}} \end{pmatrix}^{\mathsf{T}}.$$

Caution: $-q$ describes the same orientation.

- Be able to change quaternion multiplication to matrix/vector multiplication fluently.

$$p \odot q = \begin{pmatrix} p_0 q_0 - p_v \cdot q_v \\ p_0 q_v + q_0 p_v + p_v \times q_v \end{pmatrix} = p^{\mathsf{L}} q = q^{\mathsf{R}} p,$$

where

$$p^{\mathsf{L}} \triangleq \begin{pmatrix} p_0 & -p_v^{\mathsf{T}} \\ p_v & p_0 \mathcal{I}_3 + [p_v \times] \end{pmatrix}, \qquad q^{\mathsf{R}} \triangleq \begin{pmatrix} q_0 & -q_v^{\mathsf{T}} \\ q_v & q_0 \mathcal{I}_3 - [q_v \times] \end{pmatrix}.$$

$$
\begin{aligned}
\bar{x}^{\mathrm{u}} &= \left(q^{\mathrm{uv}}\right)^{\mathsf{L}} \left(q^{\mathrm{vu}}\right)^{\mathsf{R}} \bar{x}^{\mathrm{v}} \\
&= \begin{pmatrix} q_0 & -q_v^{\mathsf{T}} \\ q_v & q_0 \mathcal{I}_3 + [q_v \times] \end{pmatrix} \begin{pmatrix} q_0 & q_v^{\mathsf{T}} \\ -q_v & q_0 \mathcal{I}_3 + [q_v \times] \end{pmatrix} \begin{pmatrix} 0 \\ x^{\mathrm{v}} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0_{1\times 3} \\ 0_{3\times 1} & q_v q_v^{\mathsf{T}} + q_0^2 \mathcal{I}_3 + 2 q_0 [q_v \times] + [q_v \times]^2 \end{pmatrix} \begin{pmatrix} 0 \\ x^{\mathrm{v}} \end{pmatrix}.
\end{aligned}
$$

Compare with

$$
R^{\mathrm{uv}}(n^{\mathrm{v}}, \alpha) = \mathcal{I}_3 - \sin\alpha [n^{\mathrm{v}} \times] + (1 - \cos\alpha) [n^{\mathrm{v}} \times]^2. \tag{3.17}
$$

it can be recognized that if we choose

$$
q^{\mathrm{uv}}(n^{\mathrm{v}}, \alpha) = \begin{pmatrix} \cos \frac{\alpha}{2} \\ -n^{\mathrm{v}} \sin \frac{\alpha}{2} \end{pmatrix},
$$

the two rotation formulations are equivalent since

$$\begin{aligned}\bar{x}^{\mathrm{u}} &= \begin{pmatrix} 1 & 0_{1\times 3} \\ 0_{3\times 1} & \mathcal{I}_3 - 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}[n^{\mathrm{v}}\times] + 2\sin^2\frac{\alpha}{2}[n^{\mathrm{v}}\times]^2 \end{pmatrix} \begin{pmatrix} 0 \\ x^{\mathrm{v}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0_{1\times 3} \\ 0_{3\times 1} & \mathcal{I}_3 - \sin\alpha[n^{\mathrm{v}}\times] + (1-\cos\alpha)\,[n^{\mathrm{v}}\times]^2 \end{pmatrix} \begin{pmatrix} 0 \\ x^{\mathrm{v}} \end{pmatrix}. \end{aligned} \tag{3.31}$$

Here, we made use of standard trigonometric relations and the fact that since $\|n^{\mathrm{v}}\|_2 = 1$, $n^{\mathrm{v}}\left(n^{\mathrm{v}}\right)^{\mathsf{T}} = \mathcal{I}_3 + [n^{\mathrm{v}}\times]^2$. Hence, it can be concluded that q^{uv} can be expressed in terms of α and n^{v} as in (3.30).

Equivalently, $q^{\mathrm{uv}}(n^{\mathrm{v}}, \alpha)$ can also be written as

$$q^{\mathrm{uv}}(n^{\mathrm{v}}, \alpha) = \exp(-\tfrac{\alpha}{2}\bar{n}^{\mathrm{v}}) = \sum_{k=0}^{\infty} \tfrac{1}{k!} \left(-\tfrac{\alpha}{2}\bar{n}^{\mathrm{v}}\right)^k, \tag{3.32}$$

where

$$(\bar{n}^{\mathrm{v}})^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^\mathsf{T}, \tag{3.33a}$$

$$(\bar{n}^{\mathrm{v}})^1 = \begin{pmatrix} 0 & (n^{\mathrm{v}})^\mathsf{T} \end{pmatrix}^\mathsf{T}, \tag{3.33b}$$

$$(\bar{n}^{\mathrm{v}})^2 = \bar{n}^{\mathrm{v}} \odot \bar{n}^{\mathrm{v}} = \begin{pmatrix} -\|n^{\mathrm{v}}\|_2^2 & 0_{3\times 1} \end{pmatrix}^\mathsf{T} = \begin{pmatrix} -1 & 0_{3\times 1} \end{pmatrix}^\mathsf{T}, \tag{3.33c}$$

$$(\bar{n}^{\mathrm{v}})^3 = \begin{pmatrix} 0 & -(n^{\mathrm{v}})^\mathsf{T} \end{pmatrix}^\mathsf{T}, \tag{3.33d}$$

This leads to

$$\begin{aligned} q^{\mathrm{uv}}(n^{\mathrm{v}}, \alpha) &= \exp(-\tfrac{\alpha}{2}\bar{n}^{\mathrm{v}}) = \sum_{k=0}^{\infty} \tfrac{1}{k!} \left(-\tfrac{\alpha}{2}\bar{n}^{\mathrm{v}}\right)^k \\ &= \begin{pmatrix} 1 - \frac{1}{2!}\frac{\alpha^2}{4} + \frac{1}{4!}\frac{\alpha^4}{16} - \dots \\ -\frac{\alpha}{2}n^{\mathrm{v}} + \frac{1}{3!}\frac{\alpha^3}{8}n^{\mathrm{v}} - \frac{1}{5!}\frac{\alpha^5}{32}n^{\mathrm{v}} + \dots \end{pmatrix} \\ &= \begin{pmatrix} \cos\frac{\alpha}{2} \\ -n^{\mathrm{v}}\sin\frac{\alpha}{2} \end{pmatrix}. \end{aligned} \tag{3.34}$$

Note the similarity to (3.18) and (3.19). The reason why both rotation matrices and unit quaternions can be described in terms of an exponential of a rotation vector will be discussed in §3.3.1.

Cautions

- Some exiting estimation algorithms typically assume that the unknown states and parameters are represented in Euclidean space.
  - For instance, they assume that the subtraction of two orientations gives information about the difference in orientation and that the addition of two orientations is again a valid orientation.
- If you work in a parameterizing space, you still need to be careful.
  - For instance, due to wrapping and gimbal lock, subtraction of Euler angles and rotation vectors can result in large numbers even in cases when the rotations are similar.
- Subtraction of unit quaternions and rotation matrices do not in general result in a valid rotation.
- The equality constraints on the norm of unit quaternions and on the determinant and the orthogonally of rotation matrices are typically hard to include in the estimation algorithms.
- We will discuss some methods later to represent orientation in estimation algorithms that deals with the issues described above.
  - frequently used correct representations and algorithms

- we will also discuss some alternative methods to parametrize orientation for estimation purposes.

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{C(t+\delta t) + C(t)}{\delta t}$$

Since $C(t+\delta t)$ also represents a rotation, we choose to write is as

$$C(t+\delta t) C(t) A(t)$$

for some rotation matrix $A(t)$.

Recall that rotations about the x, y, and z axes can be written respectively as

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \qquad R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad R_z = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_x R_y R_z$$

for some ϕ, θ, and φ, which are then known as Tait–Bryan angles. Multiplying R out yields

$$\begin{bmatrix} \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta\cos\varphi - \sin\varphi\cos\phi & \sin\phi\sin\varphi\sin\theta + \cos\phi\cos\varphi & \sin\phi\cos\theta \\ \sin\phi\sin\varphi + \sin\theta\cos\phi\cos\varphi & -\sin\phi\cos\varphi + \sin\varphi\sin\theta\cos\phi & \cos\phi\cos\theta \end{bmatrix}.$$

If ϕ, θ, and φ approach zero, we can make a small angle approximation yielding

$$R = R_x R_y R_z = \begin{bmatrix} 1 & \varphi & -\theta \\ -\varphi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix}.$$

Since δt is small, we can write

$$A(t) = I + \delta\Psi(t)$$

where

$$\delta\Psi(t) = \begin{bmatrix} 0 & -\delta\varphi & \delta\theta \\ \delta\varphi & 0 & -\delta\phi \\ -\delta\theta & \delta\phi & 1 \end{bmatrix}.$$

Thus, substituting into our original forward-difference yields

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}t} &= \lim_{\delta t \to 0} \frac{C(t)(I + \delta\Psi(t)) - C(t)}{\delta t} \\ &= C(t) \lim_{\delta t \to 0} \frac{\mathrm{d}\Psi}{\mathrm{d}t} \\ &= C(t)\Omega(t) \end{aligned}$$

where

$$\Omega(t) = \begin{bmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{bmatrix}.$$

This is the skew-symmetric form of the angular velocity vector $\boldsymbol{\omega}(t)$, which we can acquire periodically from the gyroscope. Thus, we are interested in solving the differential equation

$$\dot{C}(t) = C(t)\Omega(t).$$

This has the solution

$$C(t) = C(0) \cdot \exp\left(\int_0^t \Omega(t)\ \mathrm{d}t \right).$$

Correct ways we use

- Recall the set of rotations is a Lie group, so there exists an exponential map from a corresponding Lie algebra.
- We use exponential map from so(3) to SO(3) and logarithm from SO(3) to so(3).
- We represent orientations on SO(3) using unit quaternions or rotation matrices,
- We represent orientation deviations using rotation vectors on R3 (Key! This mimic how we deal with rotation in R2 using Euler angle.)

Specifically,

we encode an orientation q_t^{nb} in terms of a *linearization point* parametrized either as a unit quaternion $\tilde{q}_t^{\text{nb}}$ or as a rotation matrix $\tilde{R}_t^{\text{nb}}$ and an *orientation deviation* using a rotation vector η_t. Assuming that the orientation deviation is expressed in the body frame b,[1]

$$q_t^{\text{nb}} = \tilde{q}_t^{\text{nb}} \odot \exp\left(\frac{\bar{\eta}_t^{\text{b}}}{2}\right), \qquad R_t^{\text{nb}} = \tilde{R}_t^{\text{nb}} \exp\left([\eta_t^{\text{b}} \times]\right), \tag{3.35}$$

where analogously to (3.34) and (3.19),

$$\exp(\bar{\eta}) = \begin{pmatrix} \cos \|\eta\|_2 \\ \frac{\eta}{\|\eta\|_2} \sin \|\eta\|_2 \end{pmatrix}, \tag{3.36a}$$

$$\exp([\eta \times]) = \mathcal{I}_3 + \sin\left(\|\eta\|_2\right) \left[\tfrac{\eta}{\|\eta\|_2} \times\right] + \left(1 - \cos\left(\|\eta\|_2\right)\right) \left[\tfrac{\eta}{\|\eta\|_2} \times\right]^2. \tag{3.36b}$$

Gyroscope measurement models (Later)

As discussed in §2.2, the gyroscope measures the angular velocity $\omega_{\mathrm{ib}}^{\mathrm{b}}$ at each time instance t. However, as shown in §2.4, its measurements are corrupted by a slowly time-varying bias $\delta_{\omega,t}$ and noise $e_{\omega,t}$. Hence, the gyroscope measurement model is given by

$$y_{\omega,t} = \omega_{\mathrm{ib},t}^{\mathrm{b}} + \delta_{\omega,t}^{\mathrm{b}} + e_{\omega,t}^{\mathrm{b}}. \tag{3.41}$$

$$
{}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where θ is the angle of rotation. If we compose two such rotations, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, through θ_1 and θ_2 respectively, the product is given by: