Lecture 8 Part 3: From Directional Derivative to Covariant Derivative

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Math 178:
Nonlinear Data Analysis

Directional Derivatives

Definition

Let f be a differentiable real-valued function on $\mathbf{E}^3$, and let $\mathbf{v}_p$ be a tangent vector to $\mathbf{E}^3$. Then the number

$$\mathbf{v}_p[f] = \frac{d}{dt}(f(\mathbf{p} + t\mathbf{v}))|_{t=0}$$

is called the *derivative of f with respect to $\mathbf{v}_p$*.

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Note

U_1, U_2, U_3 is the standard frame field on $\mathbf{E}^3$, then $U_i[f] = \partial f / \partial x_i$.

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U_1, U_2, U_3 is the standard frame field on $\mathbf{E}^3$, then $U_i[f] = \partial f / \partial x_i$.

Lemma

If $\mathbf{v}_p = (v_1, v_2, v_3)_p$ *is a tangent vector to* $\mathbf{E}^3$, *then*

$$\mathbf{v}_p[f] = \sum v_i \frac{\partial f}{\partial x_i}(\mathbf{p} = (v \cdot \nabla f)(p).$$

Directional Derivatives

Theorem

Let f and g be functions on $\mathbf{E}^3$, $\mathbf{v}_p$ and $\mathbf{w}_p$ tangent vectors, a and b numbers. Then

1. $(a\mathbf{v}_p + b\mathbf{w}_p)[f] = a\mathbf{v}_p[f] + b\mathbf{w}_p[f]$.
2. $\mathbf{v}_p[af + bg] = a\mathbf{v}_p[f] + b\mathbf{v}_p[g]$.
3. $\mathbf{v}_p[fg] = \mathbf{v}_p[f] \cdot g(\mathbf{p}) + f(\mathbf{p}) \cdot \mathbf{v}_p[g]$.

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Corollary

If V and W are vector fields on $\mathbf{E}^3$ and f, g, h are real-valued functions, then

1. $(fV + gW)[h] = fV[h] + gW[h]$.
2. $V[af + bg] = aV[f] + bV[g]$ *for all real numbers a and b.*
3. $V[fg] = V[f] \cdot g + f \cdot V[g]$.

Covariant Derivatives on $\mathbf{E}^3$

Definition

Let W be a vector field on $\mathbf{E}^3$ and let $\mathbf{v}$ be a tangent vector to $\mathbf{E}^3$ at the point $\mathbf{p}$. Then the *covariant derivative* of W with respect to $\mathbf{v}$ is the tangent vector

$$\nabla_v W = W(\mathbf{p} + t\mathbf{v})'(0)$$

at the point $\mathbf{p}$.

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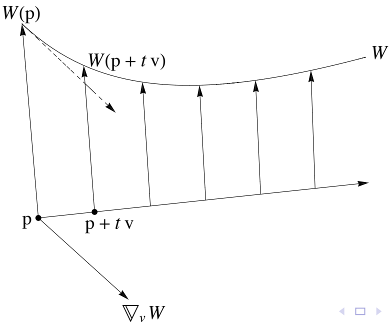
Evidently

$\nabla_v W$ measures the initial rate of change of $W(\mathbf{p})$ as $\mathbf{p}$ moves in the $\mathbf{v}$ direction.



Covariant Derivatives

Example

Suppose $W = x^2 U_1 + yz U_3$, and $\mathbf{v} = (-1, 0, 2)$ at $\mathbf{p} = (2, 1, 0)$.

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Lemma

If $W = \sum w_i U_i$ is a vector field on $\mathbf{E}^3$, and $\mathbf{v}$ is a tangent vector at $\mathbf{p}$, then

$$\nabla_v W = \sum \mathbf{v}[w_i] U_i(\mathbf{p}).$$

In short, to apply ∇_v to a vector field, apply $\mathbf{v}$ to its Euclidean coordinates.

Covariant Derivatives

Theorem

Let $\mathbf{v}$ *and* $\mathbf{w}$ *be tangent vectors to* $\mathbf{E}^3$ *at* $\mathbf{p}$, *and let* Y *and* Z *be vector fields on* $\mathbf{E}^3$. *Then*

1. $\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$ *for all numbers* a *and* b.
2. $\nabla_v(aY + bZ) = a\nabla_v Y + b\nabla_v A$ *for all numbers* a *and* b.
3. $\nabla_v(fY) = \mathbf{v}[f]Y(\mathbf{p}) + f(\mathbf{p})\nabla_v Y$ *for all (differentiable) functions* f.
4. $\mathbf{v}[Y \cdot Z] = \nabla_v Y \cdot Z(\mathbf{p}) + Y(\mathbf{p}) \cdot \nabla_v Z$.

Covariant Derivatives

Note

We can define $\nabla_V W$ naturally, where V is also a vector field.

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Corollary

Let V, W, Y, and Z be vector fields on $\mathbf{E}^3$. Then

1. $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$ *for all numbers a and b.*
2. $\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_W Y$ *for all functions f and g.*
3. $\nabla_V(fY) = V[f]Y + f\nabla_V Y$ *for all functions f.*
4. $V[Y \cdot Z] = \nabla_V Y \cdot Z + Y \cdot \nabla_V Z$.

Covariant Derivative on a Surface

Motivation
We want to systematically study the intrinsic geometry of a surface.

Key
We need to generalize Gauss's idea

$$dN_p : T_p(S) \to T_p(S),$$

where N is a normal vector field.

How
We must find a way of differentiating a vector field with respect to a direction, and this must be compatible with differentiation in a Euclidean space.

Covariant Derivative on a Surface

Definition

Let w be a differentiable vector field in an open set $U \subset S$ and $p \in U$. Let $y \in T_p(S)$. Consider a parametrized curve

$$\alpha : (-\epsilon, \epsilon) \to U,$$

with $\alpha(0) = p$ and $\alpha'(0) = y$, and let $w(t)$, $t \in (-\epsilon, \epsilon)$, be the restriction of the vector field w to the curve α. The vector field obtained by the normal projection of $w'(0)$ onto the plane $T_p(S)$ is called the *covariant derivative* at p of the vector field w relative to the vector y. This covariant derivative is denoted by $(Dw/dt)(0)$ or $(D_y w)(p)$.

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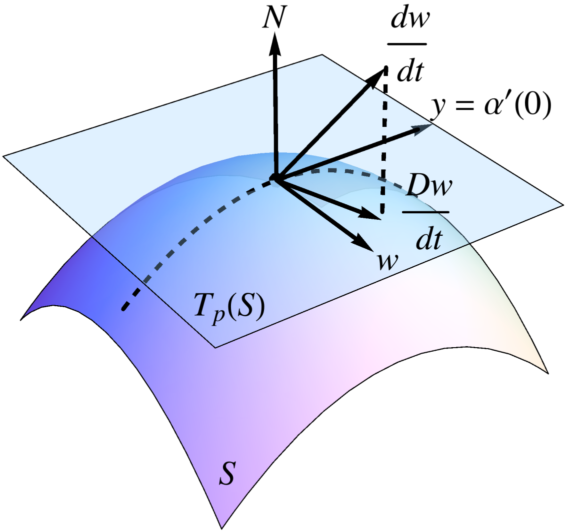
Where:

A *(tangent) vector field* in an open set $U \subset S$ of a regular surface S is a correspondence w that assigns to each $p \in U$ a vector $w(p) \in T_p(S)$. The vector field w is *differentiable* at p if, for some parametrization $\mathbf{x}(u, v)$ in p, the components a and b of $w = a(u, v)\mathbf{x}_u + b(u, v)\mathbf{x}_v$ in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ are differentiable functions at p. w is differentiable in U if it is differentiable for every $p \in U$.


N
dw/dt
y = α′(0)
Dw/dt
w
T_p(S)
S

The Covariant Derivative in Local Coordinates

The above definition makes use of the normal vector of S and of a particular curve α, tangent to y at p. To show that covariant differentiation is a concept of the intrinsic geometry and that it does not depend on the choice of the curve α, we shall obtain its expression in terms of a parametrization $\mathbf{x}(u, v)$ of S in p.

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$$\begin{aligned} \frac{Dw}{dt} &= (a' + \Gamma^1_{11} a u' + \Gamma^1_{12} a v' + \Gamma^1_{12} b u' + \Gamma^1_{22} b v')\mathbf{x}_u \\ &\quad + (b' + \Gamma^2_{11} a u' + \Gamma^2_{12} a v' + \Gamma^2_{12} b u' + \Gamma^2_{22} b v')\mathbf{x}_v. \end{aligned} \tag{1}$$

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$$\begin{aligned}\frac{Dw}{dt} &= (a' + \Gamma^1_{11} au' + \Gamma^1_{12} av' + \Gamma^1_{12} bu' + \Gamma^1_{22} bv')\mathbf{x}_u \\ &\quad + (b' + \Gamma^2_{11} au' + \Gamma^2_{12} av' + \Gamma^2_{12} bu' + \Gamma^2_{22} bv')\mathbf{x}_v. \qquad (1)\end{aligned}$$

This expression shows that Dw/dt depends only on the vector $(u', v') = y$ and not on the curve α. Furthermore, the surface makes its appearance in Eq. **??** through the Christoffel symbols, that is, through the first fundamental form. Our assertions are, therefore, proved.

The Covariant Derivative in Local Coordinates

Note

The definition of covariant derivative of a vector field is the analogue for surfaces of the usual differentiation of vectors in the plane.

The Covariant Derivative in Local Coordinates

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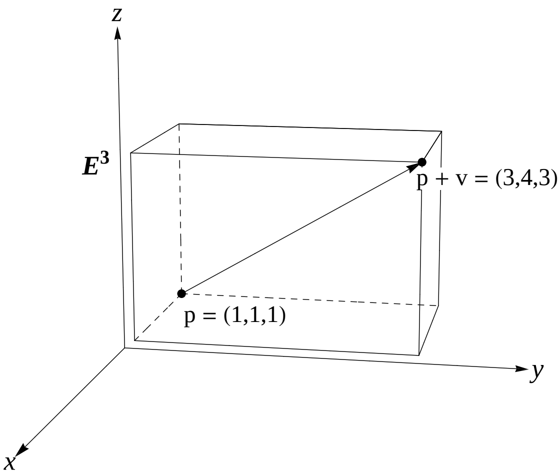
From the local expression of Dw/dt, we see that if S is a plane, we know that it is possible to find a parametrization in such a way that $E = G = 1$ and $F = 0$. A quick inspection of the equations that give the Christoffel symbols shows that in this case the Γ_{ij}^k become zero. In this case, it follows from Eq. **??** that the covariant derivative agrees with the usual derivative of vectors in the plane (this can also be seen geometrically from the definition). The covariant derivative is, therefore, a generalization of the usual derivative of vectors in the plane.

Appendix: Vector Fields on $\mathbf{E}^3$

Definition

A *tangent vector* $\mathbf{v}_p$ to $\mathbf{E}^3$ consists of two points of $\mathbf{E}^3$: its *vector part* $\mathbf{v}$ and its *point of application* $\mathbf{p}$.



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Definition

Let $\mathbf{p}$ be a point of $\mathbf{E}^3$. The set $T_p(\mathbf{E}^3)$ consisting of all tangent vectors that have $\mathbf{p}$ as point of application is called the *tangent space* of $\mathbf{E}^3$ at $\mathbf{p}$.

Fig. 1.3

Note

We define $\mathbf{v}_p + \mathbf{w}_p$ to be $(\mathbf{v} + \mathbf{w})_p$, and if c is a number we define $c(\mathbf{v}_p)$ to be $(c\mathbf{v})_p$.

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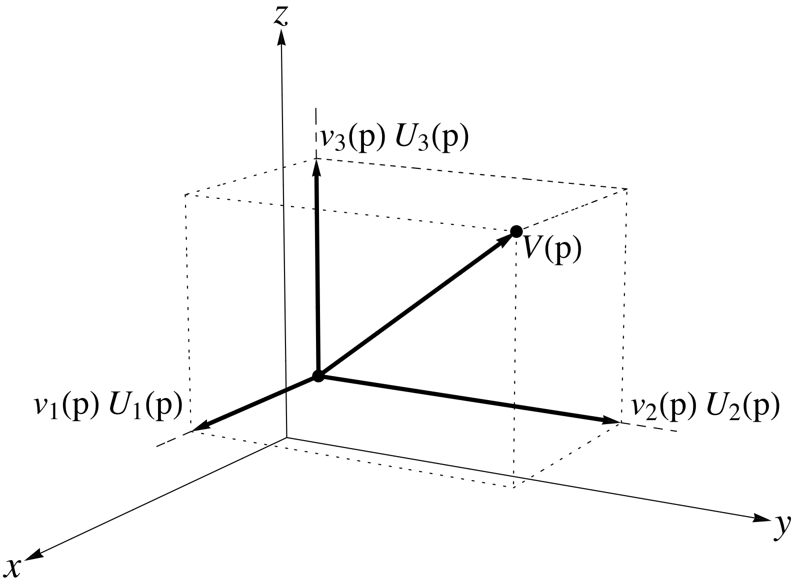
Lemma

If V is a vector field on $\mathbf{E}^3$, there are three uniquely determined real-valued functions v_1, v_2, v_3 on $\mathbf{E}^3$ such that

$$V = v_1 U_1 + v_2 U_2 + v_3 U_3.$$

The functions v_1, v_2, v_3 are called the Euclidean coordinate functions of V.


z
$v_3(\text{p})\ U_3(\text{p})$
$V(\text{p})$
$v_1(\text{p})\ U_1(\text{p})$
$v_2(\text{p})\ U_2(\text{p})$
y
x

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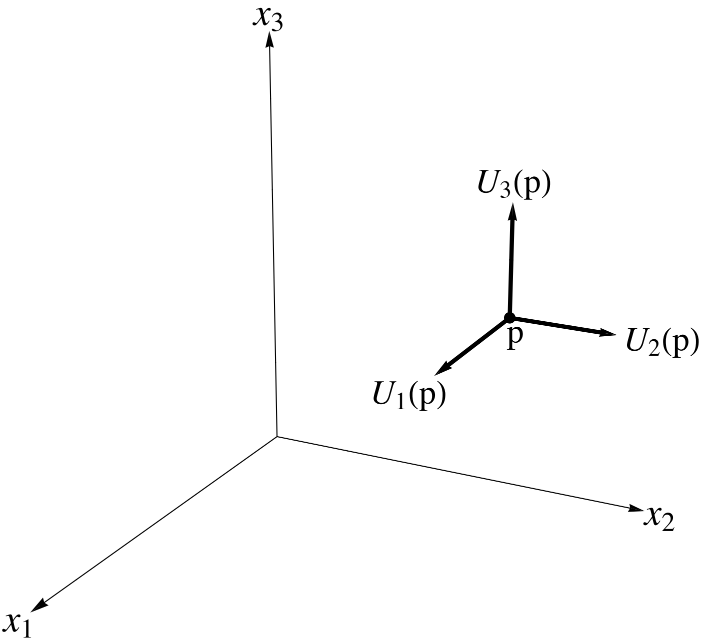
Definition

Let U_1, U_2, and U_3 be the vector fields on $\mathbf{E}^3$ such that

$$U_1(\mathbf{p}) = (1, 0, 0)_p$$
$$U_2(\mathbf{p}) = (0, 1, 0)_p$$
$$U_3(\mathbf{p}) = (0, 0, 1)_p$$

for each point $\mathbf{p}$ of $\mathbf{E}^3$. We call U_1, U_2, U_3—collectively—the *natural frame field* on $\mathbf{E}^3$.


x_3
$U_3(\mathrm{p})$
p
$U_2(\mathrm{p})$
$U_1(\mathrm{p})$
x_2
x_1

Appendix: Vector Fields on $\mathbf{E}^3$

Definition

Vector fields E_1, E_2, E_3 on $\mathbf{E}^3$ constitute a *frame field* on $\mathbf{E}^3$ probided

$$E_i \cdot E_j = \delta_{ij} \qquad (1 \leq i, j \leq 3)$$

where δ_{ij} is the Kronecker delta.

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where δ_{ij} is the Kronecker delta.

The term *frame field* is justified by the fact that at each point $\mathbf{p}$ the three vectors $E_1(\mathbf{p}), E_2(\mathbf{p}), E_3(\mathbf{p})$ form a frame at $\mathbf{p}$. We anticipated this by calling U_1, U_2, U_3 the natural frame field on $\mathbf{E}^3$.

Appendix: Vector Fields on $\mathbf{E}^3$

Example: The Cylindrical Frame Field

Let r, θ, z be the usual cylindrical coordinate functions on $\mathbf{E}^3$. We shall pick a unit vector field in the direction in which each coordinate increases (when the other two are held constant).

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$$E_1 = \cos\theta U_1 + \sin\theta U_2,$$

pointing straight out from the z axis.

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$$E_1 = \cos\theta U_1 + \sin\theta U_2,$$

pointing straight out from the z axis. Then

$$E_2 = -\sin\theta U_1 + \cos\theta U_2$$

points in the direction of increasing θ.

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pointing straight out from the z axis. Then

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points in the direction of increasing θ. Finally the direction of increase of z is, of course, straight up, so

$$E_3 = U_3.$$

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pointing straight out from the z axis. Then

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points in the direction of increasing θ. Finally the direction of increase of z is, of course, straight up, so

$$E_3 = U_3.$$

It is easy to check that $E_i \cdot E_j = \delta_{ij}$, so this is a frame field (defined on all of $\mathbf{E}^3$ except the z axis). We call it the *cylindrical frame field* on $\mathbf{E}^3$.

Appendix: Vector Fields on $\mathbf{E}^3$

Example: The Spherical Frame Field

Let E_1, E_2, E_3 be the cylindrical frame field. For spherical coordinates, the unit vector field F_2 in the direction of increasing θ is the same as above, so $F_2 = E_2$.

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Let E_1, E_2, E_3 be the cylindrical frame field. For spherical coordinates, the unit vector field F_2 in the direction of increasing θ is the same as above, so $F_2 = E_2$. The unit vector field F_1, in the direction of increasing ρ, points straight out from the origin; hence it may be expressed as

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$$F_1 = \sin\varphi E_1 + \cos\varphi E_3.$$

Similarly, the vector field for increasing φ is

$$F_3 = -\cos\varphi E_1 + \sin\varphi E_3.$$

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$$F_1 = \sin\varphi E_1 + \cos\varphi E_3.$$

Similarly, the vector field for increasing φ is

$$F_3 = -\cos\varphi E_1 + \sin\varphi E_3.$$

Thus the formulas for E_1, E_2, E_3 yield

$$\begin{aligned} F_1 &= \sin\varphi(\cos\theta U_1 + \sin\theta U_2) + \cos\varphi U_3 \\ F_2 &= -\sin\theta U_1 + \cos\theta U_2 \\ F_3 &= -\cos\varphi(\cos\theta U_1 + \sin\theta U_2) + \sin\varphi U_3. \end{aligned}$$

Appendix: Vector Fields on Surfaces

Definition

A *vector field* w in an open set $U \subset S$ of a regular surface S is a correspondence which assigns to each $p \in U$ a vector $w(p) \in T_p(S)$. The vector field w is differentiable at $p \in U$ if, for some parametrization $\mathbf{x}(u, v)$ at p, the functions $a(u, v)$ and $b(u, v)$ given by

$$w(p) = a(u, v)\mathbf{x}_u + b(u, v)\mathbf{x}_v$$

are differentiable functions at p; it is clear that this definition does not depend on the choice of $\mathbf{x}$.

Appendix: Vector Fields on Surfaces

Compare

A *vector field* in an open set $U \subset \mathbb{R}^2$ is a map which assigns to each $q \in U$ a vector $w(q) \in \mathbb{R}^2$. The vector field w is said to be *differentiable* if writing $q = (x, y)$ and $w(q) = (a(x, y), b(x, y))$, the functions a and b are differentiable functions in U.

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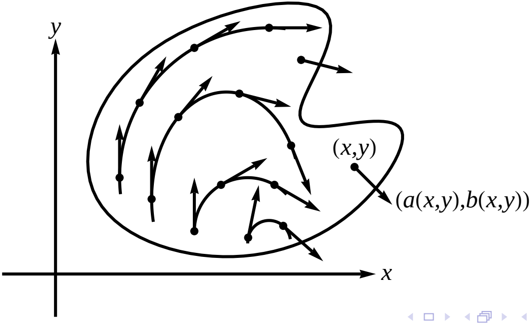
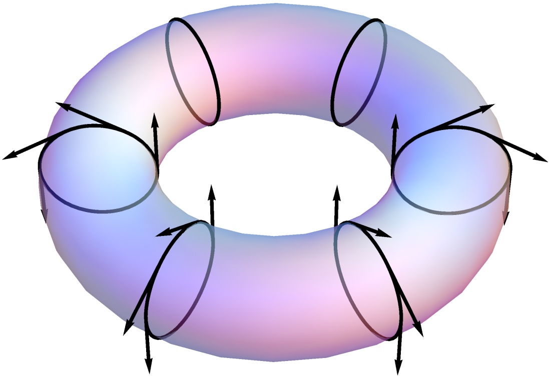
Geometrically, the definition corresponds to assigning to each point $(x, y) \in U$ a vector with coordinates $a(x, y)$ and $b(x, y)$ which vary differentiably with (x, y).



Appendix: Vector Fields on Surfaces

Example

A vector field in the usual torus T is obtained by parametrizing the meridians of T by arc length and defining $w(p)$ as the velocity vector of the meridian through p. Notice that $\|w(p)\| = 1$ for all $p \in T$. It is left as an exercise to verify that w is differentiable.



Appendix: Vector Fields on Surfaces

Example

A similar procedure, this time on the sphere S^2 and using the semimeridians of S^2, yields a vector field w defined in the sphere minus the two poles N and S. To obtain a vector field defined in the whole sphere, reparametrize all the semimeridians by the same parameter t, $-1 < t < 1$, and define $v(p) = (1 - t^2)w(p)$ for $p \in S^2 \setminus \{N, S\}$ and $v(N) = v(S) = 0$.