Lecture 9

Math 178

Nonlinear Data Analytics

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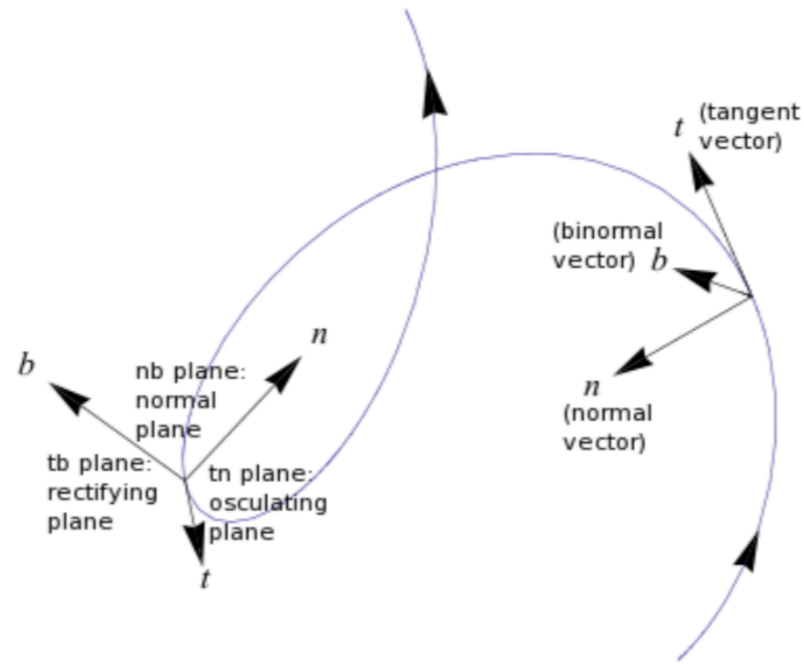
Recall: Key Characteristics by Using Moving Frames

- For curves: Frenet frame and formul

$$\alpha'(s) \stackrel{\triangle}{=} t(s)$$
$$\alpha''(s) = k(s)n(s)$$
$$t(s) \wedge n(s) = b(s)$$

$$\begin{cases} t' = kn, \\ n' = -kt - \tau b, \\ b' = \tau n \end{cases}$$



Key: Express the rate change of the frame in the same frame!
The coefficients involved are the important characteristics.

Fundamental Theorem of the Local Theory of Curves

Theorem

Given differentiable functions $k(s) > 0$ and $\tau(s), s \in I$, there exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ such that s is the arc length, $k(s)$ is the curvature, and $\tau(s)$ is the torsion of α Moreover, any other curve $\overline{\alpha}$ satisfying the same conditions differs from α by a rigid motion; that is, there exists an orthogonal map ρ of $\mathbb{R}^3$, with positive determinant, and a vector c such that $\overline{\alpha} = \rho \circ \alpha + c$.

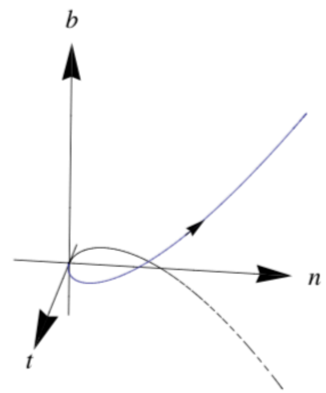
Curvature and Torsion Locally totally determine a curve: Local Canonical Form

Let us now take the system $0xyz$ in such a way that the origin 0 agrees with $\alpha(0)$ and that $t = (1,0,0)$, $n = (0,1,0)$, and $b = (0,0,1)$. Under these conditions, $\alpha(s) = (x(s), y(s), z(s))$ is given by

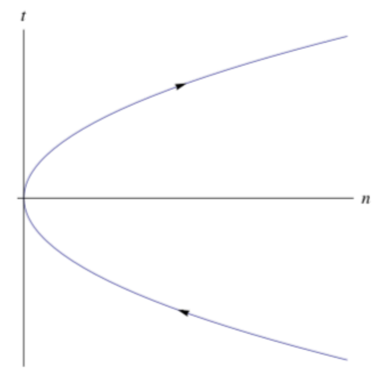
$$\begin{cases} x(s) = s - \dfrac{k^2 s^3}{6} + R_x, \\ y(s) = \dfrac{ks^2}{2} + \dfrac{k' s^3}{6} + R_y, \\ z(s) = -\dfrac{k\tau s^3}{6} + R_z, \end{cases} \tag{1}$$

where $R = (R_x, R_y, R_z)$. The representation (1) is called the *local canonical form* of α, in a neighborhood of $s = 0$.

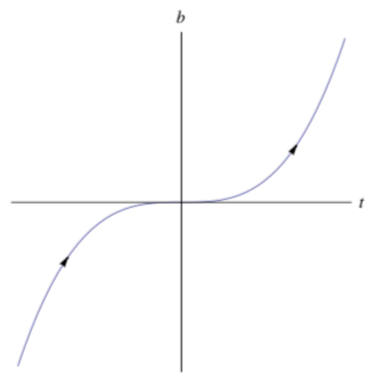
A Sketch of projections of the trace of α, for small *s*, in the *tn*, *tb*, and *nb* planes:



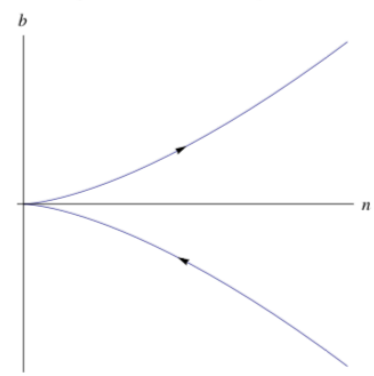
A Curve in $\mathbb{R}^3$



Projection over the plane *tn*



Projection over the plane *tb*



Projection over the plane *nb*

For Surfaces: Christofel Symbols are basic characteristics!

- Recall:

Trihedron at a Point of a Surface

S will denote, as usual, a regular, orientable, and oriented surface. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization in the orientation of S. It is possible to assign to each point of $\mathbf{x}(U)$ a natural trihedron given by the vectors $\mathbf{x}_u$, $\mathbf{x}_v$, and N.

By expressing the derivatives of the vectors $\mathbf{x}_u$, $\mathbf{x}_v$, and N in the basis $\{\mathbf{x}_u, \mathbf{x}_v, N\}$, we obtain

$$
\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + L_1 N, \\
\mathbf{x}_{uv} &= \Gamma^2_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + L_2 N, \\
\mathbf{x}_{vu} &= \Gamma^1_{21}\mathbf{x}_u + \Gamma^2_{21}\mathbf{x}_v + \bar{L}_2 N, \\
\mathbf{x}_{vv} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + L_3 N, \\
N_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \\
N_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v.
\end{aligned}
$$

For the Lie group SE(3) as the configuration of all rigid body motions: Gyroscope data are basic characteristics!

- What Does the Gyroscope data measure?

The gyroscope measures the angular velocity of the body frame with respect to the inertial frame, expressed in the body frame [147], denoted by $\omega_{\mathrm{ib}}^{\mathrm{b}}$. This angular velocity can be expressed as

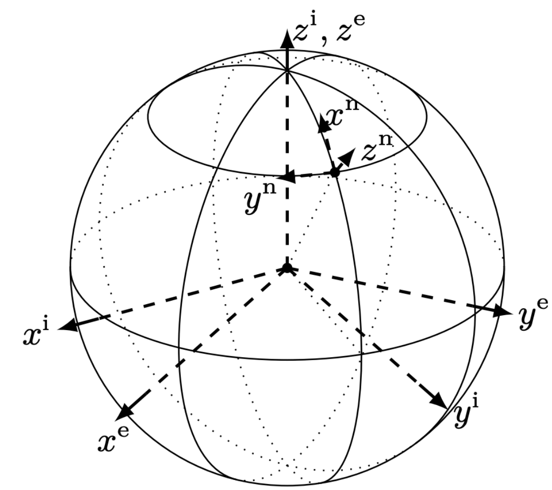
$$\omega_{\mathrm{ib}}^{\mathrm{b}} = R^{\mathrm{bn}} \left(\omega_{\mathrm{ie}}^{\mathrm{n}} + \omega_{\mathrm{en}}^{\mathrm{n}} \right) + \omega_{\mathrm{nb}}^{\mathrm{b}}, \tag{2.2}$$

where R^{bn} is the rotation matrix from the navigation frame to the body frame. The *earth rate*, *i.e.* the angular velocity of the earth frame with respect to the inertial frame is denoted by ω_{ie}. The earth rotates around its own z-axis in 23.9345 hours with respect to the stars [101]. Hence, the earth rate is approximately $7.29 \cdot 10^{-5}$ rad/s.

In case the navigation frame is not defined stationary with respect to the earth, the angular velocity ω_{en}, *i.e.* the *transport rate* is non-zero. The angular velocity required for navigation purposes — in which we are interested when determining the orientation of the body frame with respect to the navigation frame — is denoted by ω_{nb}.

Note: Also express the angular velocity of the body frame in the body frame which is a moving frame!

What Does the accelerometer data measure?



The accelerometer measures the specific force f in the body frame b [147]. This can be expressed as

$$f^{\mathrm{b}} = R^{\mathrm{bn}}(a^{\mathrm{n}}_{\mathrm{ii}} - g^{\mathrm{n}}), \tag{2.3}$$

where g denotes the gravity vector and $a^{\mathrm{n}}_{\mathrm{ii}}$ denotes the linear acceleration of the sensor expressed in the navigation frame, which is

$$a^{\mathrm{n}}_{\mathrm{ii}} = R^{\mathrm{ne}} R^{\mathrm{ei}} a^{\mathrm{i}}_{\mathrm{ii}}. \tag{2.4}$$

The subscripts on the linear acceleration a are used to indicate in which frame the differentiation is performed. For navigation purposes, we are interested in the position of the sensor in the navigation frame p^{n} and its derivatives as performed in the navigation frame

$$\tfrac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{n}}\big|_{\mathrm{n}} = v^{\mathrm{n}}_{\mathrm{n}}, \qquad \tfrac{\mathrm{d}}{\mathrm{d}t} v^{\mathrm{n}}\big|_{\mathrm{n}} = a^{\mathrm{n}}_{\mathrm{nn}}. \tag{2.5}$$

A relation between a_{ii} and a_{nn} can be derived by using the relation between two rotating coordinate frames. Given a vector x in a coordinate frame u,

$$\frac{\mathrm{d}}{\mathrm{d}t} x^{\mathrm{u}}\Big|_{\mathrm{u}} = \frac{\mathrm{d}}{\mathrm{d}t} R^{\mathrm{uv}} x^{\mathrm{v}}\Big|_{\mathrm{u}} = R^{\mathrm{uv}} \frac{\mathrm{d}}{\mathrm{d}t} x^{\mathrm{v}}\Big|_{\mathrm{v}} + \omega^{\mathrm{u}}_{\mathrm{uv}} \times x^{\mathrm{u}}, \tag{2.6}$$

where $\omega^{\mathrm{u}}_{\mathrm{uv}}$ is the angular velocity of the v-frame with respect to the u-frame, expressed in the u-frame. For a derivation of this relation in the context of inertial navigation, see [59, 147]. For a general introduction, see any textbook on dynamics, *e.g.* [92, 96].

Using the fact that

$$p^{\mathrm{i}} = R^{\mathrm{ie}} p^{\mathrm{e}}, \tag{2.7}$$

the velocity v_{i} and acceleration a_{ii} can be expressed as

$$v^{\mathrm{i}}_{\mathrm{i}} = \frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{i}}\Big|_{\mathrm{i}} = \frac{\mathrm{d}}{\mathrm{d}t} R^{\mathrm{ie}} p^{\mathrm{e}}\Big|_{\mathrm{i}} = R^{\mathrm{ie}} \frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{e}}\Big|_{\mathrm{e}} + \omega^{\mathrm{i}}_{\mathrm{ie}} \times p^{\mathrm{i}} = v^{\mathrm{i}}_{\mathrm{e}} + \omega^{\mathrm{i}}_{\mathrm{ie}} \times p^{\mathrm{i}}, \tag{2.8a}$$

$$\begin{aligned} a^{\mathrm{i}}_{\mathrm{ii}} &= \frac{\mathrm{d}}{\mathrm{d}t} v^{\mathrm{i}}_{\mathrm{i}}\Big|_{\mathrm{i}} = \frac{\mathrm{d}}{\mathrm{d}t} v^{\mathrm{i}}_{\mathrm{e}}\Big|_{\mathrm{i}} + \frac{\mathrm{d}}{\mathrm{d}t} \omega^{\mathrm{i}}_{\mathrm{ie}} \times p^{\mathrm{i}}\Big|_{\mathrm{i}} \\ &= a^{\mathrm{i}}_{\mathrm{ee}} + 2\omega^{\mathrm{i}}_{\mathrm{ie}} \times v^{\mathrm{i}}_{\mathrm{e}} + \omega^{\mathrm{i}}_{\mathrm{ie}} \times \omega^{\mathrm{i}}_{\mathrm{ie}} \times p^{\mathrm{i}}, \end{aligned} \tag{2.8b}$$

where we have made use of (2.5), (2.6), and the fact that the angular velocity of the earth is constant, *i.e.* $\frac{\mathrm{d}}{\mathrm{d}t}\omega_{\mathrm{ie}}^{\mathrm{i}} = 0$. Using the relation between the earth frame and the navigation frame

$$p^{\mathrm{e}} = R^{\mathrm{en}} p^{\mathrm{n}} + n_{\mathrm{ne}}^{\mathrm{e}}, \tag{2.9}$$

where n_{ne} is the distance from the origin of the earth coordinate frame to the origin of the navigation coordinate frame, expressions similar to (2.8) can be derived. Note that in general it can not be assumed that $\frac{\mathrm{d}}{\mathrm{d}t}\omega_{\mathrm{en}} = 0$. Inserting the obtained expressions into (2.8), it is possible to derive the relation between a_{ii} and a_{nn}. Instead of deriving these relations, we will assume that the navigation frame is fixed to the earth frame, and hence R^{en} and $n_{\mathrm{ne}}^{\mathrm{e}}$ are constant and

$$v_{\mathrm{e}}^{\mathrm{e}} = \left.\frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{e}}\right|_{\mathrm{e}} = \left.\frac{\mathrm{d}}{\mathrm{d}t} R^{\mathrm{en}} p^{\mathrm{n}}\right|_{\mathrm{e}} = R^{\mathrm{en}} \left.\frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{n}}\right|_{\mathrm{n}} = v_{\mathrm{n}}^{\mathrm{e}}, \tag{2.10a}$$

$$a_{\mathrm{ee}}^{\mathrm{e}} = \left.\frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{e}}^{\mathrm{e}}\right|_{\mathrm{e}} = \left.\frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{n}}^{\mathrm{e}}\right|_{\mathrm{n}} = a_{\mathrm{nn}}^{\mathrm{e}}. \tag{2.10b}$$

This is a reasonable assumption as long as the sensor does not travel over significant distances as compared to the size of the earth and it will be one of the model assumptions that we will use in this tutorial. More on the modeling choices will be discussed in Chapter 3.

Inserting (2.10) into (2.8) and rotating the result, it is possible to express $a^{\mathrm{n}}_{\mathrm{ii}}$ in terms of $a^{\mathrm{n}}_{\mathrm{nn}}$ as

$$a^{\mathrm{n}}_{\mathrm{ii}} = a^{\mathrm{n}}_{\mathrm{nn}} + 2\omega^{\mathrm{n}}_{\mathrm{ie}} \times v^{\mathrm{n}}_{\mathrm{n}} + \omega^{\mathrm{n}}_{\mathrm{ie}} \times \omega^{\mathrm{n}}_{\mathrm{ie}} \times p^{\mathrm{n}}, \tag{2.11}$$

where a_{nn} is the acceleration required for navigation purposes. The term $\omega^{\mathrm{n}}_{\mathrm{ie}} \times \omega^{\mathrm{n}}_{\mathrm{ie}} \times p^{\mathrm{n}}$ is known as the *centrifugal acceleration* and $2\omega^{\mathrm{n}}_{\mathrm{ie}} \times v^{\mathrm{n}}_{\mathrm{n}}$ is known as the *Coriolis acceleration*. The centrifugal acceleration is typically absorbed in the (local) gravity vector. In Example 2.2, we illustrate the magnitude of both the centrifugal and the Coriolis acceleration.

Example 2.2 (Magnitude of centrifugal and Coriolis acceleration) *The centrifugal acceleration depends on the location on the earth. It is possible to get a feeling for its magnitude by considering the property of the cross product stating that*

$$\|\omega_{ie}^n \times \omega_{ie}^n \times p^n\|_2 \leq \|\omega_{ie}^n\|_2 \|\omega_{ie}^n\|_2 \|p^n\|_2. \tag{2.12}$$

Since the magnitude of ω_{ie} *is approximately* $7.29 \cdot 10^{-5}$ rad/s *and the average radius of the earth is* 6371 km *[101], the magnitude of the centrifugal acceleration is less than or equal to* $3.39 \cdot 10^{-2}$ $\mathrm{m/s^2}$.

The Coriolis acceleration depends on the speed of the sensor. Let us consider a person walking at a speed of 5 km/h. *In that case the magnitude of the Coriolis acceleration is approximately* $2.03 \cdot 10^{-4}$ $\mathrm{m/s^2}$. *For a car traveling at* 120 km/h, *the magnitude of the Coriolis acceleration is instead* $4.86 \cdot 10^{-3}$ $\mathrm{m/s^2}$.

Important concept: Intrinsic Geometry

- Only depend on the first fundamental form for the surface.
- In general, only depend on the Riemannian metric.

The coefficients Γ_{ij}^k, $i, j, k = 1, 2$, are called the *Christoffel symbols* of S in the parametrization $\mathbf{x}$. Since $\mathbf{x}_{uv} = \mathbf{x}_{vu}$, we conclude that $\Gamma_{12}^1 = \Gamma_{21}^1$ and $\Gamma_{12}^2 = \Gamma_{21}^2$; that is, the Christoffel symbols are symmetric relative to the lower indices.

To determine the Christoffel symbols, we take the inner product of the first four relations with $\mathbf{x}_u$ and $\mathbf{x}_v$, obtaining the system

$$\begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \frac{1}{2} E_u, \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = F_u - \frac{1}{2} E_v, \end{cases}$$

$$\begin{cases} \Gamma_{12}^1 E + \Gamma_{12}^2 F = \langle \mathbf{x}_{uv}, \mathbf{x}_u \rangle = \frac{1}{2} E_v, \\ \Gamma_{12}^1 F + \Gamma_{12}^2 G = \langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle = \frac{1}{2} G_u, \end{cases}$$

$$\begin{cases} \Gamma_{22}^1 E + \Gamma_{22}^2 F = \langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle = F_v - \frac{1}{2} G_u, \\ \Gamma_{22}^1 F + \Gamma_{22}^2 G = \langle \mathbf{x}_{vv}, \mathbf{x}_v \rangle = \frac{1}{2} G_v. \end{cases}$$

E, F, G are the coefficient of the first fundamental form.

- What is the first fundamental form?
- See slides on Lecture 9 Part 2.
- Key: E, F, G are determined by the Riemannian metric.

What is a Riemannian metric?

Overview of Riemannian Metric

A vector space V^n + $\langle , \rangle$ = Inner product space

(or Euclidean space if $n < +\infty$)

A regular surface S + First Fundamental form = Riemannian Surface

($\langle , \rangle_{\mathbb{R}^3} \Downarrow \langle , \rangle_p$ on T_pS)

A differentiable mfld M + A Riemannian metric

= Riemannian mfld

- Riemannian metric on M
- M and N isometric (or local isometric)
- Examples of Riem. mflds
- Using metric to define "length", "volume".

Review for Inner product on R^n and isometry of regular surfaces

Formally, an inner product space is a vector space V over the field F together with an *inner product*, i.e., with a map

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

that satisfies the following three axioms for all vectors $x, y, z \in V$ and all scalars $a \in F$

- Conjugate symmetry:
  $$\langle x, y \rangle = \overline{\langle y, x \rangle}.$$
  Note that when $F = \mathbf{R}$, conjugate symmetry reduces to symmetry.
- Linearity in the first argument:
  $$\langle ax, y \rangle = a\langle x, y \rangle.$$
  $$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle.$$
- Positive-definiteness:
  $$\langle x, x \rangle \geq 0$$
  $$\langle x, x \rangle = 0 \Rightarrow x = 0$$

DEFINITION 1. *A diffeomorphism* $\varphi : S \longrightarrow \bar{S}$ *is an* isometry *if for all* $p \in S$ *and all pairs* $w_1, w_2 \in T_p(S)$ *we have*

$$\langle w_1, w_2 \rangle_p = \langle d\varphi_p(w_1), d\varphi_p(w_2) \rangle_{\varphi(p)}.$$

Riemannina Metrics

2.1 DEFINITION. A *Riemannian metric* (or *Riemannian structure*) on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $\langle\ ,\ \rangle_p$ (that is, a symmetric, bilinear, positive-definite form) on the tangent space T_pM, which varies differentiably in the following sense: If $\mathbf{x}: U \subset \mathbf{R}^n \to M$ is a system of coordinates around p, with $\mathbf{x}(x_1, x_2, \ldots, x_n) = q \in \mathbf{x}(U)$ and $\frac{\partial}{\partial x_i}(q) = d\mathbf{x}_q(0, \ldots, 1, \ldots, 0)$, then $\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \rangle_q = g_{ij}(x_1, \ldots, x_n)$ is a differentiable function on U.

It is usual to delete the index p in the function $\langle\ ,\ \rangle_p$ whenever there is no possibility of confusion. The function g_{ij} $(= g_{ji})$ is called *the local representation of the Riemannian metric* (or "the g_{ij} of the metric") *in the coordinate system* $\mathbf{x}: U \subset \mathbf{R}^n \to M$. A differentiable manifold with a given Riemannian metric will be called a *Riemannian manifold*.

2.2 DEFINITION. Let M and N be Riemannian manifolds. A diffeomorphism $f: M \to N$ (that is, f is a differentiable bijection with a differentiable inverse) is called an *isometry* if:

$$(1) \quad \langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}, \text{ for all } p \in M,\ u, v \in T_pM.$$

* If there exists an isometry $f: M \to N$, then M and N are said to be isometric.

2.3 DEFINITION. Let M and N be Riemannian manifolds. A differentiable mapping $f: M \to N$ is a *local isometry* at $p \in M$ if there is a neighborhood $U \subset M$ of p such that $f: U \to f(U)$ is a diffeomorphism satisfying (1).

It is common to say that a Riemannian manifold M is *locally isometric* to a Riemannian manifold N if for every p in M there exists a neighborhood U of p in M and a local isometry $f: U \to f(U) \subset N$.

Example: The Gaussian Curvature can be written as an algebraic combination of E, F, G and their derivatives!
Therefore, K is determined by the Riemannian metric!
So Gaussian Curvature is an intrinsic characteristic!

$$\begin{aligned}(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12} &= -E\frac{eg - f^2}{EG - F^2} \\ &= -EK. \qquad (1)\end{aligned}$$